A CHARACTERIZATION OF THE BEHRENS RADICAL

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1. Introduction

In 1954 E. A. Behrens [3] introduced a radical class B lying properly between the Jacobson radical class J and the Brown-McCoy radical class G. Behrens [3] defined the B-radical. $B(R)$. of a ring R as follows: $B(R)=\{x\in R\colon y\in(y^2-y)$ for all $y\in(x)\}$, where (y^2-y) and (x) denote the principal ideals of R generated by the elements y^2-y and x, respectively. N. J. Dwinsky in [4] presented B as the upper radical class $\mathfrak{S}(M)$ determined by the special class M=all subdirectly irreducible rings with idempotent hearts such that the hearts contain non-zero idempotent elements. In this paper we give a somewhat simpler characterization of the Behrens radical class.

We shall employ the following notation throughout.

$H(R)$ denotes all homomorphic images of the ring R.

$\mathscr{H}(R)$ denotes the heart of the ring R.

$I\leq R$ denotes I is an ideal of the ring R.

$I\lneqq R$ denotes $I\leq R$ but $I\neq R$.

$R\approx R'$ denotes the rings R and R' are isomorphic.

0, depending upon the context in which it appears, denotes the ring 0, the ideal 0, or the class $\{0\}$.

We shall use the following characterization of radical classes [1, p. 105].

A subclass P of a universal class W of rings is a radical class if and only if P satisfies the following three conditions.

(i) P is homomorphically closed.

(ii) If $\{I_\alpha\colon \alpha\in\Gamma\}$ is a chain of P-ideals of a ring $R\in W$, then $\bigcup_{\alpha\in\Gamma} I_\alpha$ is a P-ideal of R.

(iii) If $R\in W$ and if $I\leq R$ such that $I\in P$ and $R/I\in P$, then $R\in P$.

Let W be a universal class of rings and define a subclass B^* of W by $B^*=\{R\in W\colon R$ has no homomorphic image with non-zero idempotent elements$\}$.

2. Theorems

THEOREM 1. *The class B^* is a radical class.*

PROOF. We shall show that each of the conditions (i), (ii), (iii) is satisfied by B^*. First let $R \in H(B^*)$ and let $g(R)$ be a non-zero homomorphic image of R. If $g(R)$ contained a non-zero idempotent element, then so would the homomorphic image of some ring in B^*. Thus R must be in B^* and hence $B^*=H(B^*)$.

To show that B^* satisfies condition (ii), let $\{I_\alpha\}$ be any chain of B^*-ideals of a ring $R \in W$. By way of contradiction assume that $\bigcup I_\alpha \notin B^*$. Then there exists a homomorphism f and an element $x \in \bigcup I_\alpha$ such that $f(x)=e=e^2\neq 0$. Now $x \in I_\alpha$ for some α, and $e=f(x) \in f(I_\alpha) \in H(B^*)=B^*$. Thus $f(I_\alpha)$ is a non-zero homomorphic image of $I_\alpha \in B^*$ and $f(I_\alpha)$ has a non-zero idempotent element—a contradiction to the assumption that $I_\alpha \in B^*$. Hence $\bigcup I_\alpha \in B^*$ and condition (ii) is satisfied.

Finally let $R \in W$, $I \leq R$, $I \in B^*$, $R/I \in B^*$. We must show that $R \in B^*$. Again, by way of contradiction, assume that $R \notin B^*$. Then let f be a homomorphism and let $0\neq e^2=e \in f(R)$. Now $e \notin f(I)$, because $I \in B^*$. So $f(R)/f(I)$ contains the non-zero idempotent element $e+f(I)$. But then R/I may be mapped homomorphically onto the ring $f(R)/f(I)$ by the homomorphism $\hat{f}$ defined by $\hat{f}(x+I)=f(x)+f(I)$, while $R/I \in B^*$. This contradiction forces R to be a member of B^*.

THEOREM 2. *The radical class B^* is hereditary.*

PROOF. Let $R \in B^*$ and let $0\neq I \lneq R$. If $I \notin B^*$, then I has a homomorphic image I/K with a non-zero idempotent element $x+K$, i.e., $K \triangleleft I$, $x \in I$, $x \notin K$, $x^2-x \in K$. Now $x^2 \notin K$; for if $x^2 \in K$, then $-x \in K$ and hence $x \in K$. Let K' denote the ideal of R generated by K. By [2, p. 186] we have $(K')^3 \subseteq K$. Thus if $x \in K'$, then $x^3 \in (K')^3 \subseteq K$. Since $x \in I$ and $x^2-x \in K \subseteq I$, we have $x^3-x^2 \in xK \subseteq K$. But then $x^2 \in K$. Thus $x \notin K'$. But $x^2-x \in K \subseteq K'$ and $x \notin K'$ imply that R/K' has a non-zero idempotent element $x+K'$. This is contrary to the assumption that $R \in B^*$. Hence we must have $I \in B^*$ and thus that B^* is hereditary.

LEMMA 1. [4, Lemma 74]. *Let P be a hereditary radical. Then a subdirectly irreducible ring R with heart $\mathscr{H}(R)$ is P-semi-simple if and only if $\mathscr{H}(R)$ is P-semi-simple.*

LEMMA 2. $B^*\subseteq B$.

PROOF. Since every ring in M is B-semi-simple and since B is the upper radical determdined by the class M it suffices to show that every ring in M is B^*-semi-simple. For this let $R\in M$. Then $0\neq\mathscr{H}(R)$ and $\mathscr{H}(R)$ contains a non-zero idempotent element. Thus $B^*(\mathscr{H}(R))=0$, and so by Lemma $B^*(R)=0$.

NOTE. Let E be the class of all rings whose hearts have non-zero idempotent elements. Then $B^* \subseteq\mathfrak{S}(E)$, the upper radical determined by the class E. Clealy, since $M\subseteq E$ we have $\mathfrak{S}(E)\subseteq\mathfrak{S}(M)=B$.

THEOREM 3. $B^*=B$.

PROOF. Let $R\in B$ and assume that $R\neq B^*(R)$. Now $R\notin B^*$ implies that $R/B^*(R)$ is not zero. Since $R/B^*(R)$ is B^*-semi-simple, $R/B^*(R)$ has a non-zero homomorphic image R/I with a non-zero idempotent element $x+I(B^*(R)\subseteq I,\ x^2-x\in I,\ x\notin I)$. But $R\in B$, so we must have $x\in(x^2-x)\subseteq I$. This is a contradiction. Hence $R\in B$ implies $R\in B^*$, i.e., $B\subseteq B^*$. By Lemma 2 $B^*=B$

THEOREM 4. *Let R be a ring. Then $B^*(R)=\bigcup\{I\colon I\leq R$ and every non-zero ideal of R/I can be mapped homomorphically onto a ring with a non-zero idempotent element$\}$.*

PROOF. Since $R/B^*(R)$ is B^*-semi-simple, each non-zero ideal of $R/B^*(R)$ is B^*-semi-simple. Therefore each non-zero ideal of $R/B^*(R)$ can be mapped homomorphically onto a ring with a non-zero idempotent element. Now let I be any ideal of R such that each non-zero ideal of R/I can be mapped homomorphiclly onto a ring with a non-zero idempotent element. We show that $B^*(R)\subseteq I$. By way of contradiction assume that $B^*(R)\subsetneq I$. Then $0\neq(B^*(R)+I)/I\leq R/I$ and hence $(B^*(R)+I)/I$ can be mapped homomorphically onto a ring with a non-zero idempotent element. But $(B^*(R)+I)/I\approx B^*(R)/B^*(R)\cap I\in B^*$, i.e., $B^*(R)/B^*(R)\cap I$ can be mapped homorphically onto a ring with a non-zero idempotent element. This is a contradition, because $B^*(R)/B^*(R)\cap I\in B^*$. Hence $B^*(R)\subseteq I$ and so $B^*(R)\subseteq\bigcap\{I\colon I\leq R$ and every non-zero ideal of R/I can be mapped homomorphically onto a ring with a non-zero idempotent element$\}$. But $B^*(R)$ is such an ideal I and so equality obtains.

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