NOTE ON A GENERALIZATION OF SAALSCHÜTZ'S THEOREM

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1. Preliminaries.

Making use of the familiar abbreviation

$$(\lambda)_m=\frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}=\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+m-1),$$

we write the power series definition of the generalized hypergeometric ${}_pF_q$ function in the form

$${}_pF_q\left[\begin{matrix}a_1,\cdots, & a_p; \\ b_1,\cdots, & b_q;\end{matrix}\; z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!}. \tag{1}$$

For the usual restrictions on the b parameters and the conditions of convergence of the general series (1) see Slater [4, p. 45].

In a recent paper published in the Mathematische Zeitschrift Kalla and Saxena [3, p. 234] have given what they call a generalization of the well-known Saalschütz's theorem (cf., e. g., [4], p. 49)

$${}_3F_2\left[\begin{matrix}a,\ b,\ -n; \\ c, 1+a+b-c-n;\end{matrix}\; 1\right]=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \tag{2}$$

where n is a non-negative integer. The formula obtained by Kalla and Saxena is [3, p. 234]

$${}_3F_2\left[\begin{matrix}\alpha,\ \beta+m,\ \frac{\lambda}{\mu}; \\ \beta,\ \frac{\lambda}{\mu}+1;\end{matrix}\; 1\right]=\frac{\left(\beta-\frac{\lambda}{\mu}\right)_m\Gamma(1-\alpha)\Gamma\left(1+\frac{\lambda}{\mu}\right)}{(\beta)_m\Gamma\left(1-\alpha+\frac{\lambda}{\mu}\right)}, \tag{3}$$

where $\mathrm{Re}\,(1-\alpha)>m$, $m=0, 1, 2, \cdots; \mu>0$ and $\beta\neq 0, -1, -2, \cdots$. Note that the special case $m=n=-\alpha$ of (3) corresponds to Saalschütz's formula (2).

The proof of (3) by the earlier writers uses certain operators of fractional integration involving hypergeometric functions. It may be of interest to observe that the formula (3) is an immediate consequence of the known result [1, p. 398]

$$\begin{aligned}I&\equiv\int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-m}{}_2F_1[-m,\ \beta;\gamma;x]\,dx\\ &=\frac{(\gamma-\rho)_m\Gamma(\rho)\Gamma(\beta-\gamma+1)}{(\gamma)_m\Gamma(\beta-\gamma+\rho+1)},\end{aligned} \tag{4}$$

where $m=0, 1, 2, \cdots$, $\mathrm{Re}(\rho)>0$ and $\mathrm{Re}(\beta-\gamma)>m-1$.

Indeed by Euler's transformation [4, p. 10]

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} z\right]=(1-z)^{c-a-b}{}_2F_1\left[\begin{matrix} c-a, c-b; \\ c; \end{matrix} z\right], \tag{5}$$

we have

$$I=\int_0^1 x^{\rho-1}{}_2F_1[\gamma+m, \gamma-\beta;\gamma;x]\,dx, \tag{6}$$

and on integrating the hypergeometric series in (6) term-by-term we at once get

$$I=\frac{1}{\rho}{}_3F_2\left[\begin{matrix} \gamma-\beta, \gamma+m, \rho; \\ \gamma, \rho+1; \end{matrix} 1\right], \tag{7}$$

provided $\mathrm{Re}(\rho)>0$ and $\mathrm{Re}(\beta-\gamma)>m-1$.

A comparison of the right-hand sides of (4) and (7) will yield the formula

$$ {}_3F_2\left[\begin{matrix} a, b+m, c; \\ b, c+1; \end{matrix} 1\right]=\frac{(b-c)_m\Gamma(1-a)\Gamma(1+c)}{(b)_m\Gamma(1-a+c)}, \tag{8}$$

$m=0, 1, 2, \cdots$, $\mathrm{Re}\,(1-a)>m$. $b\neq 0, -1, -2, \cdots$, $c\neq -1, -2, \cdots$; which is essentially the same as (3).

2. A transformation for ${}_3F_2$ [1].

In this section we apply the foregoing technique to derive a known transformation formula for the hypergeometric ${}_3F_2$ series with unit argument. The result thus obtained is indeed more general than the formula (8) and it finds several interesting applications which we shall discuss in the next section.

Consider the integral

$$J\equiv\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}{}_2F_1[\alpha, \beta;\gamma;x]\,dx. \tag{9}$$

A straight forward analysis gives us

$$J=\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}{}_3F_2\left[\begin{matrix} \alpha, \beta, \rho; \\ \gamma, \rho+\sigma; \end{matrix} 1\right], \tag{10}$$

provided $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\sigma)>0$ and $\mathrm{Re}(\gamma+\sigma-\alpha-\beta)>0$.

Next we transform Gauss's function in (9) by applying Euler's identity (5) and we have

$$J=\int_0^1 x^{\rho-1}(1-x)^{\gamma+\sigma-\alpha-\beta-1}{}_2F_1[\gamma-\alpha, \gamma-\beta;\gamma;x]\,dx,$$

which on term-by-term integration leads us to

$$J=\frac{\Gamma(\rho)\Gamma(\gamma+\sigma-\alpha-\beta)}{\Gamma(\gamma+\rho+\sigma-\alpha-\beta)}{}_3F_2\left[\begin{matrix} \gamma-\alpha, \gamma-\beta, \rho; \\ \gamma, \gamma+\rho+\sigma-\alpha-\beta; \end{matrix} 1\right], \tag{11}$$

where $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\sigma)>0$ and $\mathrm{Re}(\gamma+\sigma-\alpha-\beta)>0$.

From (10) and (11) we at once get the elegant formula (see also [2])

$$ {}_3F_2\left[\begin{matrix} a, b, c; \\ d, e; \end{matrix} 1\right]=\frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-c)\Gamma(d+e-a-b)}\,{}_3F_2\left[\begin{matrix} d-a, d-b, c; \\ d, d+e-a-b; \end{matrix} 1\right], \qquad (12) $$

valid, by analytic continuation, when $\mathrm{Re}\,(d+e-a-b-c)>0$ and $\mathrm{Re}\,(e-c)>0$.

It is interesting to note that the hypergeometric series on the left-hand side of (12) is Saalschützian when $d+e=1+a+b+c$.

Therefore, if we further let a or b equal $-m$, and apply Saalschütz's formula (2) to sum the series on the left, (12) will reduce immediately to the summation formula (8).

3. Applications

As a first instance of applications of the formula (12) we show that the recent summation formula (3) is equivalent to Saalschütz's theorem (2). Indeed by (12) we have

$$ {}_3F_2\left[\begin{matrix} \alpha, \beta+m, \frac{\lambda}{\mu}; \\ \beta, \frac{\lambda}{\mu}+1; \end{matrix} 1\right]=\frac{\Gamma\left(1+\frac{\lambda}{\mu}\right)\Gamma(1-\alpha-m)}{\Gamma\left(1-\alpha+\frac{\lambda}{\mu}-m\right)}\,{}_3F_2\left[\begin{matrix} \beta-\alpha, -m, \frac{\lambda}{\mu}; \\ \beta, 1-\alpha+\frac{\lambda}{\mu}-m; \end{matrix} 1\right], \qquad (13) $$

and therefore (3) reduces to its equivalent form

$$ {}_3F_2\left[\begin{matrix} \beta-\alpha, \frac{\lambda}{\mu}, -m; \\ \beta, 1-\alpha+\frac{\lambda}{\mu}-m; \end{matrix} 1\right]=\frac{(\alpha)_m\left(\beta-\frac{\lambda}{\mu}\right)_m}{(\beta)_m\left(\alpha-\frac{\lambda}{\mu}\right)_m}, \qquad (14) $$

which obviously is the same as Saalschütz's formula (2) with

$a=\beta-\alpha,\ b=\frac{\lambda}{\mu},\ c=\beta$ and $n=m$.

Conversely, (14) can be shown to imply the summation formula (3). The proof would again require an application of the transform (12).

A repeated application of the formula (12) yields the well-known result [4, p. 52]

$$ \left.\begin{aligned} {}_3F_2\left[\begin{matrix} a, b, c; \\ d, e; \end{matrix} 1\right] &= \frac{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)\Gamma(d+e-a-c)} \\ &\cdot {}_3F_2\left[\begin{matrix} d-a, e-a, d+e-a-b-c; \\ d+e-a-b, d+e-a-c \ ; \end{matrix} 1\right], \end{aligned}\right\} \qquad (15) $$

where, as before, $\mathrm{Re}(d+e-a-b-c)>0$, and $\mathrm{Re}\,(a)>0$.

The formula (15) is a generalization of Dixon's summation theorem [4, p. 52]

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ 1+a-b,\ 1+a-c; \end{matrix} 1\right]
$$

$$
=\frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma\left(1+\frac{1}{2}a-b-c\right)\Gamma(1+a-b)\ \Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)}, \tag{16}
$$

which holds when $\mathrm{Re}(1+\frac{1}{2}a-b-c)>0$.

For a direct proof of (15) based on series iteration techniques and the well-known Gauss's theorem, viz.

$$
{}_2F_1\left[\begin{matrix} a,\ b; \\ c; \end{matrix} 1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \mathrm{Re}(c-a-b)>0, \tag{17}
$$

see Slater [4, pp. 52—53]. Obviously (12) may also be proved in a similar manner.

The transformation formula (15) plays an important rôle in the theory of generalized hypergeometric series. For instance, if we let $d=\frac{1}{2}(a+b+1)$ and $e=2c$, the series on the right-hand side turns out to be well-poised and hence summable by Dixon's formula (16), and we have Watson's theorem [4, p. 54]

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ \frac{1}{2}(1+a+b),\ 2c; \end{matrix} 1\right]=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a+c\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}b+c\right)}, \tag{18}
$$

provided that the series is convergent, that is that $\mathrm{Re}(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)>0$. The second series in (15) can also be summed by Watson's formula (18) when $a+b=1$ and $d+e=1+2c$ giving us Whipple's theorem [4, p. 54]

$$
{}_3F_2\left[\begin{matrix} a,\ 1-a,\ c; \\ d,\ 1+2c-d; \end{matrix} 1\right]
$$

$$
=\frac{2^{1-2c}\pi\Gamma(d)\Gamma(1+2c-d)}{\Gamma\left(\frac{1}{2}d+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}d-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+c-\frac{1}{2}d+\frac{1}{2}a\right)\Gamma\left(1+c-\frac{1}{2}d-\frac{1}{2}a\right)}, \tag{19}
$$

which holds true if $\mathrm{Re}(c)>0$, and $d\neq 0,\ -1,\ -2,\ \cdots$.

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