ON COMPLETION OF MEASURE SPACES

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The purpose of this paper is to investigate the relationship between the completion of a measure space and the completion derived from Hopf extension.

Let $(X, \mathscr{A}, \mu)$ be a measure space. Define $\mathscr{A}' = \{E \cup A \mid E \in \mathscr{A},\ A \subset B \textit{ for some } B \in \mathscr{A} \textit{ such that } \mu(B) = 0\}$, and define μ' on $\mathscr{A}'$ by the rule $\mu'(E \cup A) = \mu(E)$. We know that $\mathscr{A}'$ is a σ-algebra of subsets of X, and that μ' is a well-defined complete measure on $\mathscr{A}'$. This measure space $(X, \mathscr{A}', \mu')$ is called the completion of $(X, \mathscr{A}, \mu)$.

If X is an arbitrary set and $\mathscr{A}$ is an algebra of subsets of X, let μ be a countably additive measure on $\mathscr{A}$. Define a set function $\bar{\mu}$ on $P(X)$, the family of all subsets of X, as follows: for $T \subset X$, let $\bar{\mu}(T) = \inf\{\sum_{n=1}^{\infty} \mu(A_n) \mid T \subset \bigcup_{n=1}^{\infty} A_n$ *and* $A_1, A_2, \cdots, A_n, \cdots \in \mathscr{A}\}$. Then by the Hopf extension theorem we know that $\bar{\mu}$ is an outer measure on $P(X)$, $\bar{\mu}$ is equal to μ on the algebra $\mathscr{A}$ and $\mathscr{A} \subset \overline{\mathscr{A}}$, where $\overline{\mathscr{A}}$ is the family of all $\bar{\mu}$-measurable subsets of X and certainly $\bar{\mu}$ is countably additive on $\overline{\mathscr{A}}$.

If $\mathscr{A}$ is a σ-algebra, then $(X, \overline{\mathscr{A}}, \bar{\mu})$ is a complete measure space since it is derived from an outer measure. We wish to show that $(Z, \mathscr{A}', \mu') = (Z, \overline{\mathscr{A}}, \bar{\mu})$ for any decomposable measure space $(Z, \mathscr{A}, \mu)$.

DEFINITION. Let $(X, \mathscr{A}, \mu)$ be a measure space. Suppose that there is a subfamily $\mathscr{F}$ of $\mathscr{A}$ with the following properties:

(i) $0 \leq \mu(F) < \infty$ for all $F \in \mathscr{F}$,

(ii) the sets in $\mathscr{F}$ are pairwise disjoint and $\bigcup \mathscr{F} = X$,

(iii) if $E \in \mathscr{A}$ and $\mu(E) < \infty$ then $\mu(E) = \sum_{F \in \mathscr{F}} \mu(E \cap F)$ where the sum is defined as the supremum of the sums $\sum_{F \in \mathscr{D}} \mu(E \cap F)$, where $\mathscr{D}$ runs through all finite subfamilies of $\mathscr{F}$.

(iv) if $S \subset X$ and $S \cap F \in \mathscr{A}$ for all $F \in \mathscr{F}$, then $S \in \mathscr{A}$.

Then $(X, \mathscr{A}, \mu)$ and μ itself are said to be *decomposable* and $\mathscr{F}$ is called a *decomposition of* $(X, \mathscr{A}, \mu)$.

LEMMA. *Notation as above. If* $(X, \mathscr{A}, \mu)$ *is a decomposable measure space with decomposition* $\mathscr{F}$, *then* $(X, \mathscr{A}', \mu')$ *is also decomposable with decomposition* $\mathscr{F}$.

PROOF. The conditions (i) and (ii) are clearly satisfied. If $E \cup A \in \mathscr{A}'$, $\mu'(E \cup A) = \mu(E) < \infty$ and $A \subset B \in \mathscr{A}$ with $\mu(B) = 0$, then $\mu'(E \cup A) = \mu(E) = \sum_{F \in \mathscr{F}} \mu(E \cap F)$

$=\sum_{F\in\mathscr{F}}\mu'((E\cap F)\cup(A\cap F)) =\sum_{F\in\mathscr{F}}\mu'((E\cup A)\cap F)$. If $S\subset X$ and $S\cap F\in \mathfrak{A}'$ for all $F\in\mathscr{F}$ then $S\cap F=E_F\cup A_F$ where $E_F\in\mathfrak{A}$ and $A_F\subset B_F\in\mathfrak{A}$ and $\mu(B_F)=0$, and $S=\bigcup_{F\in\mathscr{F}}(S\cap F)=\bigcup_{F\in\mathscr{F}}(E_F\cup A_F)=\bigcup_{F\in\mathscr{F}}E_F\cup\bigcup_{F\in\mathscr{F}}A_F$.

Now $\bigcup_{F\in\mathscr{F}}A_F\subset\bigcup_{F\in\mathscr{F}}B_F\in\mathfrak{A}$ and by (iii) $\mu(\bigcup_{F\in\mathscr{F}}B_F)=\sum_{F\in\mathscr{F}}\mu(B_F)=0$, $\bigcup_{F\in\mathscr{F}}E_F\in\mathfrak{A}'$, hence $S\in\mathfrak{A}'$.

THEOREM. *Notations as above. If $(X,\mathfrak{A},\mu)$ is a decomposable measure space, then $\bar{\mathfrak{A}}=\mathfrak{A}'$ and $\bar{\mu}=\mu'$ for sets in $\bar{\mathfrak{A}}$.*

PROOF. It is clear that $\mathfrak{A}'\subset\bar{\mathfrak{A}}$. If $A\in\bar{\mathfrak{A}}$ such that $\bar{\mu}(A)=0$, then we can find a decreasing sequence $\{B_n\}$ in $\mathfrak{A}$ such that $\mu(B_n)<\frac{1}{n}$ and $B_n\supset A$ for each n. Let $B=\bigcap_{n=1}^{\infty}B_n$. Then $B\in\mathfrak{A}, A\subset B$ and $\mu(B)=0$. Hence $A\in\mathfrak{A}'$.

For any $C\in\bar{\mathfrak{A}}$ with $\bar{\mu}(C)<\infty$, we can also find a decreasing sequence $\{D_n\}$ in $\mathfrak{A}$ such that $D_n\supset C$ and $\bar{\mu}(D_n\backslash C)<\frac{1}{n}$ for each n. Hence $\bar{\mu}((\bigcap D_n)\backslash C)=\bar{\mu}(\bigcap_{n=1}^{\infty}(D_n\backslash C))=\lim_{n\to\infty}\bar{\mu}(D_n\backslash C)=0$. By the above argument we have $(\bigcap D_n)\backslash C\in\mathfrak{A}'$. Also $\bigcap D_n\in\mathfrak{A}'$. Hence $C=(\bigcap D_n)\backslash((\bigcap D_n)\backslash C)\in\mathfrak{A}'$. If $\bar{\mu}(C)=\infty$ since $(X,\mathfrak{A},\mu)$ is decomposable with decomposition $\mathscr{F}$. $C=\bigcup_{F\in\mathscr{F}}(C\cap F)$ and $C\cap F\in\bar{\mathfrak{A}}$ and $\bar{\mu}(C\cap F)<\infty$ for each F. Hence $C\cap F\in\mathfrak{A}'$ and by the lemma, $C\in\mathfrak{A}'$. Since $\mu(A)=\mu'(A)=\bar{\mu}(A)$ for any $A\in\mathfrak{A}$ and $\bar{\mathfrak{A}}=\mathfrak{A}'$, hence for any $E\cup A\in\mathfrak{A}$ with $E\in\mathfrak{A}$ and $A\subset B\in\mathfrak{A}$ with $\mu(B)=0$. Then $\bar{\mu}(E\cup A)\le\bar{\mu}(E)+\bar{\mu}(A)\le\bar{\mu}(E)+\bar{\mu}(B)=\mu(E)=\mu'(E\cup A)=\bar{\mu}(E)\le\bar{\mu}(E\cup A)$.

The following example will show that the theorem might be false if $(X,\mathfrak{A},\mu)$ is not decomposable.

EXAMPLE. Let $X=[0,1]$ and $\mathfrak{A}$ consists of all subsets $A\subset X$ such that either A or $X\backslash A$ is countable (including finite sets and the null set), and let μ be the counting measure on $\mathfrak{A}$. Then $(X,\mathfrak{A},\mu)$ is a complete measure space and $\mathfrak{A}=\mathfrak{A}'$. Let A be any uncountable subset of X such that $X\backslash A$ is also uncountable. Then $A\in\bar{\mathfrak{A}}\backslash\mathfrak{A}'$.

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REFERENCE

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