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Door Topologies on an Infinite Set

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1. **Introduction.** By definition, a *door space* is a topological space in which every subset is either open or closed. A door space X is called a *maximal door space* if it is not discrete and there does not exist a door topology properly between the discrete topology and that of X. If X is a door space whose topology cannot be properly weakened to a door topology, then X is called a *minimal door space*.

The present note is a study of door spaces with infinitely many points. Hausdorff door spaces are investigated rather extensively in Section 2. Among several structure theorems presented in the section, Theorem 1 provides us with a convenient tool in our subsequent argument. In the maximal case, Theorem 3 describes in fact all possible types of spaces which are maximal among topological spaces that are not discrete. In Section 4, we consider only T_1-spaces because it is hopable that counter examples to a standard theorem share as many properties with orthodox spaces as possible.

It seems convenient to introduce the notation we shall adopt in this paper. As usual, the cardinality of a set X is denoted by $|X|$. If X is a completely regular T_1-space, βX denotes the Stone-Čech compactification of X. If G is a decomposition of a space X, the quotient space of G is denoted by X/G. Finally, the word "point" is used in two senses so as to mean a set having only one point as well, and we use p to stand for $\{p\}$ if p is a point. This abbreviation in notation should not cause confusion, however.

2. **Structure of Hausdorff door spaces.** The purpose of this section is to seek conditions characterizing door spaces which are Hausdorff.

We begin with the following lemma.

LEMMA 1. *A Hausdorff space is a door space if and only if all save possibly one of its points are open.*

Proof. The "only if" part is a mere rephrasement of [2, 2C], while the "if" part is trivial because if there is a point p in a space X such that each point of $X-p$ is open, then $p \notin A$ implies that A is open but $p \in A$ implies that A is closed for any subset A of X.

If U is an open set containing p, then $U-p$ is called a deleted open neighborhood of p. The following version of Lemma 1 will be of frequent use later. Recall that a filter on a set is said to be free if the intersection of all members of it is void.

THEOREM 1. *A space X is a Hausdorff door space with an accumulation point p if and only if (1) $X-p$ is a discrete open subspace of X, and (2) all deleted open neighborhoods of p form a free filter on $X-p$.*

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Proof. The "if" part: Since the two conditions (1) and (2) imply that p is the only point of X which is not open, Lemma 1 ensures that it suffices to prove that X is a Hausdorff space. To do this, let x and y be any pair of distinct points in X. If none of these points coincides with p, then x and y are disjoint neighborhoods of x and y respectively. But if one of them, say y, is the point p, then (2) implies that there is a neighborhood U of y with $x \notin U$. Since $X-U$ is open by (1), $X-U$ and U are disjoint neighborhoods of x and y respectively.

The "only if" part: The point p is evidently not open, and (1) follows from Lemma 1. For (2), we observe that any set containing an open neighborhood of p is again an open neighborhood of p. Hence the collection of all open neighborhoods is identical with the neighborhood filter of p. This means, however, that deleted open neighborhoods of p form a filter $\mathcal{F}$ as every neighborhood of p must contain a point other than p. Finally, $\mathcal{F}$ is a free filter because $X-q$ is a neighborhood of p for each point q of X with $p \neq q$.

We now present another characterization of Hausdorff door spaces. Namely,

THEOREM 2. *Let X be an infinite discrete space, let K be a nonvoid compact subset of $\beta X - X$, and let G be the decomposition of $X \cup K$ with K as the only nondegenerate element. Then the quotient space $(X \cup K)/G$ is a Hausdorff door space such that the projection map g of G is a closed mapping and $g(K)$ is the only accumulation point of $(X \cup K)/G$. Conversely, any Hausdorff door space having an accumulation point is homeomorphic to a quotient space of the type described above.*

Proof. For any subset A of $X \cup K$, we have either $A = g^{-1}(g(A))$ or $A \cup K = g^{-1}(g(A))$. Hence g is a closed mapping as K must be closed in $X \cup K$, and the quotient space is a Hausdorff space. Accordingly, $(X \cup K)/G$ is a door space by Lemma 1 since $g(x)$ is evidently open for x in X. That $g(K)$ is an accumulation point follows from the fact each neighborhood of K meets X, and we have completed the proof of the direct part.

For the converse, let Y be any Hausdorff door space with accumulation point p. Then $X = Y - p$ is an infinite discrete space by Theorem 1, and Y is paracompact by [3, Lemma 1]. Hence, we may regard the inclusion map i: $X \subset Y$ as a continuous map of X into the Stone-Čech compactification βY of Y. By [2, Theorem 5.24], i extends to a unique continuous map j: $\beta X \to \beta Y$. The map j is onto because $j(X) = X$ is dense in βY. Let f denote the projection map of the *decomposition* $F = \{j^{-1}(y): y \in \beta Y\}$ *of* βX. Since the topology of the quotient space $\beta X/F$ is the largest one which makes f continuous, the unique bijective map h: $\beta X/F \to \beta Y$ defined by $j = h \cdot f$ is continuous. Moreover, h is a homeomorphism because $\beta X/F$ must be compact. Now let $K = j^{-1}(p)$, let G be the decomposition of $X \cup K$ having K as the only nondegenerate element, and let g denote the projection map of G. Since G is evidently a subcollection of F, the quotient space $(X \cup K)/G$ is the subspace $f(X \cup K)$ of $\beta X/F$ and g is identical with f cut down to $X \cup K$. Therefore, h gives a homeomorphism between $(X \cup K)/G$ and Y, because $X \cup K = j^{-1}(Y)$ This completes the proof.

3. **Maximal door spaces.** In this section, study is continued on Hausdorff door spaces by means of filter. We first characterize maximal door spaces.

THEOREM 3. *A Hausdorff space X with p as an acceuulation point is a maximal door space if and only if the collection $\mathcal{F}$ consisting of all deleted open neighborhoods of p is a free ultrafilter on $X-p$.*

Proof. The "if" part: Let A be any subset of X. If $A-p$ belongs to $\mathcal{F}$, it follows that A is either an open neighborhood or a deleted open neighborhood of p according as $p\in A$ or not. Since X is a Hausdorff space, A must be open in either case. However, if $A-p$ does not belong to $\mathcal{F}$, then maximality of $\mathcal{F}$ implies that $(X-p)-(A-p)$ is a member of $\mathcal{F}$. Accordingly, $((X-p)\cup p)-(A-p)$ is an open neighborhood of p, and $A-p$ is a closed set. Hence A is also closed because $A=(A-p)\cup p$ or $A=A-p$ according to whether p belongs to A or not, while the one point set p must be closed in the Hausdorff space X. This argument shows that X is a door space. We observe that Theorem 1 implies that $X-p$ is discrete and open. To prove the maximality of X, let X^* be the set X equipped with a topology $\mathcal{U}$ which is finer than the original topology of X. If $\mathcal{U}$ is different from the discrete topology, then p is the only accumulation point of X^* because the fact that $X-p$ is discrete and open implies that X^*-p is also discrete and open (with respect to the topology $\mathcal{U}$ of course.) Hence, all deleted open neighborhoods in X^* of p form a free filter $\mathcal{G}$ on $X^*-p=X-p$ by Theorem 1. Now, the continuity of the identity map: $X^*\to X$ implies $\mathcal{G}\supset\mathcal{F}$, and we have $\mathcal{F}=\mathcal{G}$ as $\mathcal{F}$ is an ultrafilter. This proves that $\mathcal{U}$ is the same as the original topology of X, i.e., X is a maximal space.

The "only if" part: Let X be any maximal door space having p as an accumulation point. In this case, the collection $\mathcal{F}$ of all deleted open neighborhoods of p is a free filter on $X-p$ by Theorem 1. If $\mathcal{F}$ is not an ultrafilter, let $\mathcal{G}$ be any ultrafilter on $X-p$ which contains $\mathcal{F}$. $\mathcal{G}$ is certainly a free filter because $\mathcal{F}$ is one. Hence, again by Theorem 1, we can properly expand the original topology to a Hausdorff door topology making p an accumulation point by requiring that each member of $\mathcal{G}$ be a deleted open neighborhood of p. This contradiction proves that $\mathcal{F}$ is an ultrafilter.

COROLLARY. *Every maximal door space X embeds in βD, where D is a discrete space with $|X|=|D|$.*

REMARK. The notion of maximal Hausdorff door spaces is the same as that of maximal topological spaces. This is true because if a space X is not discrete, then it has an accumulation point p and all deleted neighborhoods of p form a filter. Hence, by Theorem 1, one can expand the original topology to a Hausdorff door topology by letting $X-p$ open and discrete and by asking that every neighborhood of p with respect to the original top ology be open in the new topology.

Let $\mathcal{F}$ and $\mathcal{G}$ be filters on a set D. We say that $\mathcal{F}$ and $\mathcal{G}$ are equivalent if there is a one to one map of D onto itself which induces a one to one map of $\mathcal{F}$ onto $\mathcal{G}$; otherwise they are said to be inequivalent.

THEOREM 4. *For an infinite cardinal x there are 2^{2^x} inequivalent types of maximal door spaces of cardinality x.*

Proof. Let $X=D\cup p$ and $X'=D\cup p'$ be Hausdorff door spaces with p and p' as the only accumulation points, D being a set of any preassigned infinite cardinaitly x.

If h: $D\cup p\to D\cup p'$ is a homeomorphism, then $h(p)=p'$ and $h|D$ induces a bijection of the filter $\mathcal{F}$ of deleted open neighborhoods of p to the filter $\mathcal{F}'$ of deleted open neighborhoods of p'. Thus, by Theorem 3, it suffices to show that there are 2^{2^x} many inequivalent free ultrafilters on D. To do this, we observe that any *equivalance class of filters* on D has cardinality less than 2^{2^x}. This is true because the set of all functions from D to D is of power $x^x=2^x$. Since there are 2^{2^x} *free* ultrafilters on D by Theorem 9.2 of [1], there are 2^{2^x} many distinct equivalence classes of free ultrafilters on D. This completes the proof.

COROLLARY. *If X is an infinite discrete space, then βX has a subset Y with $|Y|=|\beta X|$ such that no pair of points in Y are equivalently embedded in βX.*

Proof. Let p be a point of $\beta X-X$, and let $\mathcal{F}_p=\{U\cap X|\ U$ is a neighborhood of p in $\beta X\}$. By Gelfand-Kolmogoroff Theorem [1], the correspondence $p\to\mathcal{F}_p$ is 1 : 1 from $\beta X-X$ to the set of all free ultrafilters on X. Hence there is a subset Y of $\beta X-X$ with cardinality $2^{2^{|X|}}$ such that if p, q are distinct points, then $\mathcal{F}_p$ and $\mathcal{F}_q$ are inequivalent as there are $2^{2^{|X|}}$ many inequivalent free ultrafilters on X by the proof of Theorem 4. In turn, this implies that $X\cup p$ is not homeomorphic with $X\cup q$. Since $|\beta X|=2^{2^{|X|}}$ and any homeomorphism of $X\cup p$ to $X\cup q$ extends to a homeomorphism of βX onto itself, this completes the proof.

We have already seen that every maximal door space embeds in βD for suitable discrete D. Although we do not know whether the converse of this is true, there do exist Hausdorff door spaces which cannot be embedded in any βD. That is, we have

THEOREM 5. *There are Hausdorff door spaces of any infinte cardinality which are not embeddable in βD for any discrete space D.*

Proof. Let x be an infinite cardinal, let Y be a discrete space with $|Y|=x$, and let Z be the one point compactification of a countably infinite discrete set. Then the topological sum of Y and Z is a door space of cardinality x but it can not be embedded in βD for any infinite discrete X. For, if $X=Y\cup Z$ is embedded in βD, so does Z. Since Z is compact and has cardinality $\aleph_0$ this contradicts to the Lemma 4 of [3].

4. Minimal door spaces. An immediate consequence of Theorem 1 is that every Hausdorff door space is totally disconnected. In non-Hausdorff cases, this is far from being true. In fact, there are many connected door spaces even in the class of T_1-spaces as we are now going to describe. We also prove that all such spaces are minimal door spaces.

LEMMA 2. *If x is an infinite cardinal, there are 2^{2^x} types of inequivalent connected T_1-spaces which are door spaces of cardinality x.*

Proof. Let X be a set with $|X|=x$, and let $\mathcal{F}$ be a free ultrafilter on X. If X is equipped with the topology having $\mathcal{F}$ as a subbase, it is easy to check that a subset A of X is open if and only if either A is empty or A is a member of $\mathcal{F}$. But if X is the sum of two members U, V of $\mathcal{F}$, then $U\cup V\neq\phi$. Hence X is connected. Since $\mathcal{F}$ is free, X is a T_1-space. Also, X is a door space because either A or $X-A$ is a member of $\mathcal{F}$. Thus we have proved that every free ultrafilter generates a topology relative to which X becomes a connected T_1-space which is a door space. Since inequivalent filters generate inequivalent topologies, the argument used in proving Theorem 4 applies in this case as well to verify the theorem.

LEMMA 3. *Any connected door space is a minimal door space*

Proof. Let $(X, \mathcal{U})$ be a connected door space and let $(X, \mathcal{V})$ be any door space. Suppose $\mathcal{U}$ is properly finer than $\mathcal{V}$. Then there is a $\mathcal{U}$-open set A which is not $\mathcal{V}$-open, i.e., a $\mathcal{U}$-open set A such that $X-B$ is $\mathcal{V}$-open. Since $\mathcal{U}$ is finer than $\mathcal{V}$, a $\mathcal{V}$-open set is $\mathcal{U}$-open and $X-A$ is $\mathcal{U}$-open. But, then $X=A\cup(X-A)$ is a separation of X with respect to the topology $\mathcal{U}$. This contradicts the connectivity of $(X, \mathcal{U})$. Therefore a connected door space is a minimal door space.

By Lemmas 2 and 3, we have many examples of minimal door spaces.

THEOREM 6. *For any infinite cardinal x, there are 2^{2^x} inequivalent minimal door spaces of cardinality x.*

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