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On Weak Extensions of a Topology

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1. Introduction. Important types of topological spaces have weak topologies with respect to some families of their subspaces. Typical examples of such spaces are k-spaces. The k-spaces are extended to c-spaces [2] or to weakly k-spaces [5], and Hausdorff k-spaces (compactly generated spaces to paracompactly generated spaces [4]. However, the common character of spaces with weak topologies with respect to some admissible properties are generalized to $\mathcal{P}$-generated spaces [4] or to hypotopological spaces [1].

In this paper, we first consider some properties of spaces with weak topologies with respect to appropriate families of subspaces and weak topologies finer than given topologies, and show that the collection of such weak extensions of a topology form a complete lattice.

2. Spaces with weak topologies with respect to some admissible families. An *admissible family* $\mathcal{A}$ of a topological space X is a collection of closed subsets of X such that every relatively closed subset of an element of $\mathcal{A}$ is also in $\mathcal{A}$. The elements of $\mathcal{A}$ will be called *$\mathcal{A}$-sets*. The space X is called *$\mathcal{A}$-generated* if and only if the original topology of X is the weak topology with respect to $\mathcal{A}$. In the literature this topology is often called the coherent topology with $\mathcal{A}$ [6].

Let $\mathcal{A}$ and $\mathcal{B}$ be two admissible families of a space X. Then clearly we have

LEMMA 2.1. *If a space X is $\mathcal{A}$-generated and $\mathcal{A}\subset\mathcal{B}$, then X is $\mathcal{B}$-generated.*

Let $\mathcal{P}$ be a property of some subsets of a space. A *$\mathcal{P}$-space* is one which possesses $\mathcal{P}$. A *$\mathcal{P}$-set* in a space X is a *closed* subset of X which possesses $\mathcal{P}$. A neighborhood which is a $\mathcal{P}$-set will be called *$\mathcal{P}$-neighborhood.* A space X is said to be *locally* $\mathcal{P}$ provided that every point of X has a $\mathcal{P}$-neighborhood.

A property $\mathcal{P}$ is called *admissible* if it is inherited by closed sets. Then clearly the collection of all $\mathcal{P}$-sets constitutes an admissible family of X. We denote this family also $\mathcal{P}$. A space X is called *$\mathcal{P}$-generated* if and only if it has the weak topology with respect to $\mathcal{P}$. Clearly locally $\mathcal{P}$-spaces are $\mathcal{P}$-generated.

In our previous work [4], elementary properties of $\mathcal{P}$-generated spaces are considered. We recall that if $\mathcal{P}$ is the compactness, then the $\mathcal{P}$-generated space is called k-space, and a paracompactly generated space is a Hausdorff $\mathcal{P}$-generated space when $\mathcal{P}$ is the paracompactness. Every compactly generated space is paracompactly generated, but the converse is not true in general.

3. Weak extensions of a topology. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{A}$ an admissible family of $(X, \mathcal{T})$. The *weak extension* of $\mathcal{T}$ with respect to $\mathcal{A}$ or simply $\mathcal{A}$-

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extension of $\mathcal{T}$ is defined to be the family $\mathcal{T}(\mathfrak{A})$ of all subsets U of X such that, for every S in $\mathfrak{A}$, $U \cap S$ is open in S. Equivalently, A is $\mathcal{T}(\mathfrak{A})$-closed if and only if $A \cap S$ is closed in S.

If $\mathcal{P}$ is an admissible property, then A is $\mathcal{T}(\mathcal{P})$-closed if and only if $A \cap S$ is a $\mathcal{T}$-$\mathcal{P}$-set for every $\mathcal{T}$-$\mathcal{P}$-set S, where "$\mathcal{T}$-$\mathcal{P}$-set" means "$\mathcal{P}$-set with respect to $\mathcal{T}$".

Clearly, $\mathcal{T}(\mathfrak{A})$ is finer than $\mathcal{T}$, and if $\mathcal{P}$ is the closedness, that is, $\mathcal{P}=\mathcal{T}$, then $\mathcal{P}=\mathcal{T}(\mathfrak{A})$. If $\mathcal{P}$ is the compactness, then the $\mathcal{P}$-extension is called the k-extension [3].

The following propositions are some elementary properties of weak extensions.

PROPOSITION 3.1. *If a subset S of $(X, \mathcal{T})$ is in $\mathfrak{A}$, then $\mathcal{T}$ agrees with $\mathcal{T}(\mathfrak{A})$ on S.*

Proof. Since S is $\mathcal{T}(\mathfrak{A})$-closed as $\mathcal{T}(\mathfrak{A})$ is finer than $\mathcal{T}$, a subset of S is $\mathcal{T}(\mathfrak{A})$-closed in the whole space whenever it is relatively $\mathcal{T}(\mathfrak{A})$-closed in S. Hence if a subset R of S is closed in S with respect to $\mathcal{T}(\mathfrak{A})$, then its intersection with any $\mathfrak{A}$-set must be $\mathcal{T}$-closed. In particular, $R=R\cap S$ is $\mathcal{T}$-closed, and $\mathcal{T}$ is finer than $\mathcal{T}(\mathfrak{A})$ on S.

PROPOSITION 3.2. *A function f from $(X, \mathcal{T})$ to a space Y is $\mathcal{T}(\mathfrak{A})$-continuous if and only if it is $\mathcal{T}$-continuous on every $\mathfrak{A}$-set of X.*

Proof. If f is $\mathcal{T}$-continuous on $\mathfrak{A}$-sets, then $(f|S)^{-1}(F)=f^{-1}(F)\cap S$ is $\mathcal{T}$-closed for F closed in Y whenever S is an $\mathfrak{A}$-set. Hence $f^{-1}(F)$ is closed with respect to $\mathcal{T}(\mathfrak{A})$, *i.e.*, f is $\mathcal{T}(\mathfrak{A})$-continuous. The converse is trivial as $\mathcal{T}=\mathcal{T}(\mathfrak{A})$ on $\mathfrak{A}$-sets by (3.1).

PROPOSITION 3.3. *$\mathcal{T}(\mathfrak{A})$ is the finest topology which agrees with $\mathcal{T}$ on $\mathfrak{A}$-sets.*

Proof. By (3.2), the identity function of X is a continuous mapping of $(X, \mathcal{T}(\mathfrak{A}))$ to $(X, \mathcal{T}')$ if $\mathcal{T}'$ agrees with $\mathcal{T}$ on $\mathfrak{A}$-sets.

For an admissible property $\mathcal{P}$ of $(X, \mathcal{T})$ which is expressed entirely in terms of set operations and closed (open) sets, we obtain the following propositions.

PROPOSITION 3.4. *If S is a $\mathcal{P}$-set with respect to $\mathcal{T}$, so is it with respect to any topology $\mathcal{T}'$ of X with*

$$\mathcal{T}\subset\mathcal{T}'\subset\mathcal{T}(\mathcal{P}).$$

Proof. Immediate from (3.1) as S must be closed with respect to $\mathcal{T}'$.

PROPOSITION 3.5. *$(X, \mathcal{T}(\mathcal{P}))$ is a $\mathcal{P}$-generated space. Consequently,*

$$(\mathcal{T}(\mathcal{P}))(\mathcal{P})=\mathcal{T}(\mathcal{P}).$$

Proof. If F intersects with all $\mathcal{T}(\mathcal{P})$-$\mathcal{P}$-sets in $\mathcal{T}(\mathcal{P})$-closed sets, so does it with all $\mathcal{T}$-$\mathcal{P}$-sets by (3.4). Since $\mathcal{T}=\mathcal{T}(\mathcal{P})$ on $\mathcal{T}$-$\mathcal{P}$-sets by (3.1), this implies that F is closed with respect to $\mathcal{T}(\mathcal{P})$.

REMARK. In a topological space $(X, \mathcal{T})$, let $\mathcal{P}$ and $\mathcal{Q}$ be two admissible properties satisfying $\mathcal{T}(\mathcal{P})\supset\mathcal{T}(\mathcal{Q})$. It is noted that in this case, a $\mathcal{T}(\mathcal{P})$-$\mathcal{P}$-set is not necessarily $\mathcal{T}(\mathcal{Q})$-$\mathcal{P}$-set.

EXAMPLE. Let $(X, \mathcal{T})$ be the one point compactification of the long line in which the point at infinity is designated by p; let $\mathcal{P}$ and $\mathcal{Q}$ denote discreteness and metrizability, respectively. In this case, $\mathcal{T}(\mathcal{P})$ is the discrete topology as every $\mathcal{P}$-set in X is a finite set by compactness. On the other hand, since a closed subspace of the long line is metrizable if and only if it is bounded above, $\mathcal{T}(\mathcal{Q})$ agrees with $\mathcal{T}$ on $X-p$ but makes the one point set p open. Therefore, all subsets of X are $\mathcal{T}(\mathcal{P})$-$\mathcal{P}$-sets, while the only $\mathcal{T}(\mathcal{Q})$-$\mathcal{P}$-sets are finite sets by countable compactness of the long line.

For admissible properties Proposition 3.3 can be generalized as follows:

PROPOSITION 3.6. *In a topological space* $(X,\mathcal{T})$, *let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be two admissible properties satisfying* $\mathcal{T}(\mathcal{P})\supset\mathcal{T}(\mathcal{Q})$. *Then* $\mathcal{T}(\mathcal{P})$ *is the finest topology which agrees with* $\mathcal{T}(\mathcal{Q})$ *on* $\mathcal{T}$-$\mathcal{P}$-*sets.*

Proof. Let $\mathcal{T}'$ be a topology which agrees with $\mathcal{T}(\mathcal{Q})$ on a $\mathcal{T}$-$\mathcal{P}$-set S. Then F is $\mathcal{T}'$-closed if and only if $F\cap S$ is $\mathcal{T}'$-closed in S, or equivalently $F\cap S$ is $\mathcal{T}(\mathcal{Q})$-closed in S by assumption. Hence F is $\mathcal{T}(\mathcal{P})$-closed since $\mathcal{T}(\mathcal{Q})\subset\mathcal{T}(\mathcal{P})$, which shows $\mathcal{T}(\mathcal{P})$ is finer than $\mathcal{T}'$.

REMARK. In each of (3.1)-(3.5), letting X be Hausdorff and $\mathfrak{A}=\mathcal{P}$ the compactness, we can obtain properties of the k-extension of a topology given in [3].

4. **The lattice of all $\mathfrak{A}$-extensions.** In this section, we are mainly concerned a collection of admissible families in a topological space $(X,\mathcal{T})$.

Let $\{\mathfrak{A}_\alpha\}$ be a collection of admissible families of $(X,\mathcal{T})$. Then clearly $\bigcup\mathfrak{A}_\alpha$ and $\bigcap\mathfrak{A}_\alpha$ are also admissible families. The elements of $\bigcup\mathfrak{A}_\alpha$ and $\bigcap\mathfrak{A}_\alpha$ are called to be $\bigvee\mathfrak{A}_\alpha$-*sets* and $\bigwedge\mathfrak{A}_\alpha$-*sets*, respectively.

LEMMA 4.1. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two admissible families of* $(X,\mathcal{T})$. *If* $\mathfrak{A}\subset\mathfrak{B}$, *then* $\mathcal{T}(\mathfrak{A})$ *is finer than* $\mathcal{T}(\mathfrak{B})$.

Proof. If F is $\mathcal{T}(\mathfrak{B})$-closed, it intersects with each $\mathfrak{A}$-set in a $\mathcal{T}$-closed set as $\mathfrak{A}$-sets are $\mathfrak{B}$-sets by hypothesis. Hence, F must be $\mathcal{T}(\mathfrak{A})$-closed as well.

LEMMA 4.2. *Let* $\{\mathfrak{A}_\alpha\}$ *be a collection of admissible families of* $(X,\mathcal{T})$. *Then* $\mathcal{T}(\bigcap\mathfrak{A}_\alpha)$ *is finer than each of* $\mathcal{T}(\mathfrak{A}_\alpha)$.

Proof. Immediate from (4.1) as each $\mathfrak{A}_\alpha$ contains $\bigcap\mathfrak{A}_\alpha$.

LEMMA 4.3. *Let* $\{\mathfrak{A}_\alpha\}$ *be a collection of admissible families of* $(X,\mathcal{T})$. *Then* $\mathcal{T}(\bigcup\mathfrak{A}_\alpha)$ *is the greatest lower bound of* $\{\mathcal{T}(\mathfrak{A}_\alpha)\}$ *in the lattice of all topologies on* X; *that is*

$$\mathcal{T}(\bigcup\mathfrak{A}_\alpha)=\bigcap\mathcal{T}(\mathfrak{A}_\alpha).$$

Proof. Since each $\mathfrak{A}_\alpha$ is contained in $\bigcup\mathfrak{A}_\alpha$, $\bigcap\mathcal{T}(\mathfrak{A}_\alpha)$ must be finer than $\mathcal{T}(\bigcup\mathfrak{A}_\alpha)$ by (4.1). To reverse this relation, let F be closed with respect to each $\mathcal{T}(\mathfrak{A}_\alpha)$ and S an arbitrary $\bigvee\mathfrak{A}_\alpha$-set. Since S must be a $\mathfrak{A}_\alpha$-set for some α, the definition of $\mathcal{T}(\mathfrak{A}_\alpha)$ for this particular choice of α implies that $F\cap S$ is closed in S with respect to $\mathcal{T}$. Hence $F\cap S$ is $\mathcal{T}$-closed in the whole space as S is a $\mathcal{T}$-closed set. This proves that F is closed with respect to $\mathcal{T}(\bigcup\mathfrak{A}_\alpha)$, and $\bigcap\mathcal{T}(\mathfrak{A}_\alpha)$ is coarser than $\mathcal{T}(\bigcup\mathfrak{A}_\alpha)$.

Let $\{\mathfrak{A}_\alpha\}$ be the collection of all admissible families of $(X, \mathcal{T})$. The partially ordered set $\{\mathcal{T}(\mathfrak{A}_\alpha)\}$ of all weak extensions of $\mathcal{T}$ contains universal bounds. In fact, $\mathcal{T}$ is the universal lower bound since the closedness is admissible and the discrete topology of X is the universal upper bound since the property of being empty is always admissible.

THEOREM 4.4. *In a topological space* $(X,\mathcal{T})$, *the collection of weak extensions* $\mathcal{T}(\mathfrak{A})$ *of* $\mathcal{T}$ *with respect to all admissible family* $\mathfrak{A}$ *is a complete lattice when ordered in the natural way.*

Proof. Any collection of $\mathfrak{A}$-extensions of $\mathcal{T}$ has an $\mathfrak{A}$-extension as g.l.b. by (4.3). Since every collection of $\mathfrak{A}$-extensions has an $\mathfrak{A}$-extension as upper bound by (4.2), it follows again by (4.3) that the g.l.b. of the collection of upper bounds (which are $\mathfrak{A}$-extensions) of any preassigned subcollection is an $\mathfrak{A}$-extension and is the l.u.b. of the given subcollection.

Let $\{\mathcal{P}_\alpha\}$ be the collection of all admissible properties of $(X,\mathcal{T})$. Then (4.1), (4.2) and (4.3) hold for $\{\mathcal{P}_\alpha\}$. Therefore we also obtain

THEOREM 4.5. *In a topological space* $(X,\mathcal{T})$, *the collection of weak extensions* $\mathcal{T}(\mathcal{P})$ *of* $\mathcal{T}$ *with respect to all admissible property* $\mathcal{P}$ *is a complete lattice when ordered in the natural way.*

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