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A Note on the Evaluation of a Definite Integral*

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1. Introduction. In a recent paper [2] the following definite integral has been evaluated:

$$\int_0^1 t^{\rho-1}(1-t)^{\beta-1}{}_2F_1\left[\begin{matrix}\lambda,\mu; \\ \beta;\end{matrix}\ 1-t\right]{}_\nu F_\sigma\left[\begin{matrix}\alpha_1, \cdots, \alpha_\nu; \\ \beta_1, \cdots, \beta_\sigma;\end{matrix}\ zt^r\right]$$

$$\cdot G\begin{matrix} p,\ q,\ k,\ r,\ l \\ A,\ [C,\ E],\ B,\ [D,\ F]\end{matrix}\left[\begin{matrix} x\,t^{m/\delta} \\ y\,t^{m/\delta}\end{matrix}\ \middle|\ \begin{matrix}(a) \\ (c);\ (e) \\ (b) \\ (d);\ (f)\end{matrix}\right]dt$$

$$=(2\pi)^{(1-\delta)M}\delta^N\frac{\Gamma[\beta]}{m^\beta}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\nu}(\alpha_j)_n}{\prod_{j=1}^{\sigma}(\beta_j)_n}\frac{z^n}{n!}$$

$$\cdot G^{p\delta+2m,\ q\delta,\ k\delta,\ r\delta,\ l\delta}_{A\delta+2m,\ [C\delta,E\delta],\ B\delta+2m,\ [D\delta,\ F\delta]}\left[\begin{matrix} x^\delta\ \delta^{(A+C-B-D)\delta} \\ y^\delta\ \delta^{(A+E-B-F)\delta}\end{matrix}\ \middle|\ \begin{matrix}\Delta[m,\ 1-\rho-\gamma_n],\ \Delta[m,\ 1-\rho-\beta-\gamma n+\lambda+\mu],\ \Delta[\delta,(a)] \\ \Delta[\delta,(c)];\ \Delta[\delta,(e)] \\ \Delta[m,\ \rho+\beta+\gamma n-\lambda],\ \Delta[m,\ \rho+\beta+\gamma n-\mu],\ \Delta[\delta,(b)] \\ \Delta[\delta,(d)];\ \Delta[\delta,(f)]\end{matrix}\right], \tag{1.1}$$

where, for the sake of brevity,

$$M=p+q+k+r+l-\frac{1}{2}(A+B+C+D+E+F),$$

$$N=\sum_{j=1}^{A}a_j+\sum_{j=1}^{D}d_j+\sum_{j=1}^{F}f_j-\sum_{j=1}^{B}b_j-\sum_{j=1}^{C}c_j-\sum_{j=1}^{E}e_j-\frac{1}{2}(A+D+F-B-C-E)+2,$$

(a) is taken to denote the sequence of A parameters

$$a_1,\ a_2,\ \cdots,\ a_j,\ \cdots,\ a_A,$$

and similarly for (b), etc.,

$\Delta[\delta,\ \lambda]$ stands for the set of δ parameters

$$\frac{\lambda}{\delta},\ \frac{\lambda+1}{\delta},\ \cdots,\ \frac{\lambda+\delta-1}{\delta},\ \delta\geqq 1,$$

and for convergence,

(i) $2(p+q+r)>A+B+C+D$

$2(p+k+l)>A+B+E+F$

$$|\arg(x)|<\pi\left[p+q+r-\frac{1}{2}(A+B+C+D)\right]$$

$$|\arg(y)|<\pi\left[p+k+l-\frac{1}{2}(A+B+E+F)\right];$$

(ii) $\nu\leqq\sigma$ (or $\nu=\sigma+1$ and $|z|<1$), $\beta_j\neq 0,\ -1,\ -2,\ -3,\ \cdots,\ j=1,\ 2,\ \cdots,\ \sigma$;

(iii) $Re(\beta)>0$, $Re(\gamma)>0$, $Re(\rho+\beta-\lambda-\mu)>0$ and

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$$Re\left[\rho+\frac{m}{\delta}(d_i+f_j)\right]>0, \quad m,\ \delta=1,\ 2,\ 3,\ \cdots,$$
$$i=1,\ 2,\ \cdots,\ r;\ j=1,\ 2,\ \cdots,\ l.$$

Here $G\begin{bmatrix} x \\ y \end{bmatrix}$ denotes a (modified) G-function of two variables discussed earlier by us (see [3] and [4]) and ${}_\nu F_\sigma[z]$ is the generalized hypergeometric function defined by [1, p. 140]

$$ {}_\nu F_\sigma\left[\begin{matrix}\alpha_1, \cdots, \alpha_\nu; \\ \\ \beta_1, \cdots, \beta_\sigma;\end{matrix}\ z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\nu}(\alpha_j)_n}{\prod_{j=1}^{\sigma}(\beta_j)_n}\frac{z^n}{n!}. \tag{1.2}$$

For the motivation of the earlier writer and for several known as well as new special forms of the integral (1.1) see [2]. Indeed his long and involved proof of the formula (1.1) makes use of a number of complicated techniques including an application of the finite difference operator

$$E_\theta f(\theta)=f(\theta+1). \tag{1.3}$$

Thus it would seem worthwhile to show, in the present note, how rapidly this formula follows by merely using Gauss's summation theorem [1, p. 3]

$$ {}_2F_1\left[\begin{matrix}\lambda, & \mu; \\ & \nu;\end{matrix}\ 1\right]=\frac{\Gamma(\nu)\Gamma(\nu-\lambda-\mu)}{\Gamma(\nu-\lambda)\Gamma(\nu-\mu)}, \tag{1.4}$$

where, for convergence,

$$Re\,(\nu-\lambda-\mu)>0.$$

Quite naturally, our approach to the problem leads us to the derivation of its obvious generalizations which we discuss rather briefly in the last section.

2. Direct proof of (1.1). In the first place we apply the multiplication formula for the double G-function to convert

$$G\left(\begin{matrix} xt^{m/\delta} \\ yt^{m/\delta}\end{matrix}\right)$$

into the more convenient form

$$G\left(\begin{matrix} x^\delta t^m \delta^{(A+C-B-D)\delta} \\ y^\delta t^m \delta^{A(+E-B-F)\delta}\end{matrix}\right),$$

where δ and m are positive integers.

Now replace the generalized hypergeometric ${}_\nu F_\sigma$ function by its power series (1.2) and the G-function by its Mellin-Barnes contour integral [4, p. 471 (1.1)], and finally, invert the order of summation and integration. This process, which can be justified fairly easily by virtue of the aforementioned conditions, leads us to the innermost definite integral in the Euler form

$$\int_0^1 t^{\rho+\gamma n+m(\xi+\eta)-1}(1-t)^{\beta-1}\,{}_2F_1\left[\begin{matrix}\lambda, \mu; \\ \beta;\end{matrix}\ 1-t\right]dt$$
$$=\sum_{s=0}^{\infty}\frac{(\lambda)_s(\mu)_s}{s!(\beta)_s}\int_0^1 t^{\rho+\gamma n-m(\xi+\eta)-1}(1-t)^{\beta+s-1}dt$$
$$=\sum_{s=0}^{\infty}\frac{(\lambda)_s(\mu)_s}{s!(\beta)_s}\frac{\Gamma[\rho+\gamma n+m(\xi+\eta)]\Gamma[\beta+s]}{\Gamma[\rho+\beta+\gamma n+m(\xi+\eta)+s]}$$

$$=\frac{\Gamma[\beta]}{\Gamma[\rho+\beta+\gamma n+m(\xi+\eta)]}\,{}_2F_1\left[\begin{matrix}\lambda,\ \mu;\\ \rho+\beta+\gamma n+m(\xi+\eta);\end{matrix}\ 1\right]$$

$$=\frac{\Gamma[\beta]\ \Gamma[\rho+\beta+\gamma n+m(\xi+\eta)-\lambda-\mu]}{\Gamma[\rho+\beta+\gamma n+m(\xi+\eta)-\lambda]\Gamma[\rho+\beta+\gamma n+m(\xi+\eta)-\mu]},$$

by Gauss's theorem (1.4), and the formula (1.1) follows when we interpret the resulting contour integral as a double G-function.

It may be of interest to observe in passing that the condition $Re\ (\gamma)>0$ stated in the set (iii) of the preceding section is all that we require of γ. Evidently this is less restrictive than that of the earlier writer [2] who assumes γ to be a positive integer.

3. Further Generalizations. The form of the formula (1.1) and the above method of its derivation would suggest the existence of its obvious extensions with the generalized hypergeometric ${}_\nu F_\sigma$ function replaced by any function $F[zt^\gamma]$ which possesses an absolutely and uniformly convergent power series expansion over the region of integration.

In general, if we let

$$H[z_1,\cdots,z_n]=\sum_{k_1,\cdots,k_n=0}^{\infty}C_{k_1,\cdots,k_n}\,z_1^{k_1}\cdots z_n^{k_n} \tag{3.1}$$

be a function of several complex variables $z_1,\ \cdots,\ z_n$ and assume that the arbitrary coefficients $C_{k_1,\cdots,k_n}$ are so chosen that the multiple series for $H[z_1t^{\gamma_1},\cdots,\ z_nt^{\gamma_n}]$ is term-by-term integrable with respect to t over the interval $[0,\ 1]$, then the method of the preceding section would apply well to the integral

$$\int_0^1 t^{\rho-1}(1-t)^{\beta-1}\,{}_2F_1\left[\begin{matrix}\lambda,\ \mu;\\ \beta;\end{matrix}\ 1-t\right]H\left[z_1t^{\gamma_1},\cdots,z_nt^{\gamma_n}\right]\,G\left[\begin{matrix}xt^{m/\delta}\\ yt^{m/\delta}\end{matrix}\right]dt, \tag{3.2}$$

whose value can be written at once by looking at the second member of (1.1).

For $n=1$, (3.2) can be specialized to involve the generalized hypergeometric function of Wright (see [7] and [8]), while in case $n=2$ we may introduce the double hypergeometric series of Kampé de Fériets' function [1, p.150] or the generalized Kampé de Fériet function defined recently by us [5, p.199]. In the general case it is of course possible to have, in the integrand of (3.2), the generalized Lauricella function of n variables which we studied (see [6], p.454) in an attempt to develop a unified theory of generalized Neumann expansions associated with hypergeometric functions of two and more complex variables.

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