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Openness Theorems on C^r Immersions and Embeddings of Some Hilbert Manifolds

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1. Introduction. In this paper, we tacitly mean manifolds by Banach manifolds (without boundary) and denote them by X, Y, and etc. The set of all C^r mappings of X into Y is denoted by $C^r(X, Y)$. An f in $C^r(X, Y)$ is said to be a C^r immersion if it is a locally C^r diffeomorphism onto a submanifold Z of Y, i.e. at each point x in X there exist open sets U of x in X and V of $f(x)$ in Y and a submanifold Z of Y such that $f|U : U \to Z \cap V$ is a C^r diffeomorphism. If $f: X \to Z$ is a C^r diffeomorphism (onto), then f is called a C^r embedding. We shall denote the set of all C^r immersions and embeddings of X into Y by $\mathcal{I}^r(X, Y)$ and $\varepsilon^r(X, Y)$ respectively.

J. McAlpin's Embedding Theorem [5] says:

(1.2) *Every second countable C^r manifold modelled on a separable Hilbert space can be C^r embedded onto a closed submanifold of a separable Hilbert space.* (Cf. [3]).

It is a generalization of Whitney's Embedding Theorem [8] :

(1.2) *There exi ts a C^r embedding (resp. immersion) of a C^r n-manifold onto a submanifold of a p-dimensional Euclidean space R^p if $p \geqq 2n+1$ (resp. $p \geqq 2n$).*

The author, by reducing (1.1), has proved a theorem [7] :

(1.3) *If X is a second countable C^r manifold on an infinite dimensional separable Hilbert space H, then there exists a C^r embedding of X onto a closed submanifold of the Hilbert space $H' = H \times R$, the cartesian product of the H and a real line R.*

This theorem (1.3) could be considered as a generalization of the above (1.2) for some desirable form.

In a case of finite dimensional manifolds, L. S. Pontrijagin [6] proved a theorem:

(1.4) *For a compact C^r n-manifold M, $\mathcal{I}^r(M, R^{2n})$ (resp. $\varepsilon^r(M, R^{2n+1})$) is open and dence in $\mathscr{C}^r(M, R^{2n})$ (resp. $\mathscr{C}^r(M, R^{2n+1})$) with the topology induced by the C^r sup. norm difined through the charts of M.*

In a case of infinite dimensional manifolds, it is natural to ask:

Are the subsets $\mathcal{I}^r(X, H')$ and $\varepsilon^r(X, H')$ open and dense in $\mathscr{C}^r(X, H')$ with any topology induced naturally?

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The purpose of this paper is to prove the following theorems answering the part for openness of the above question.

THEOREM 1. *Let X be a compact second countable C^r manifold, $r \geqq 1$, modelled on an infinite dimensional separable Hilbert space H. Then $\mathscr{S}^r(X, H')$ is a non-empty open subset of the manifold $\mathscr{C}^r(X, H')$ with the natural atlas.*

THEOREM 2. *Under the same hypothesis, $\varepsilon^r(X, H')$ is a non-empty open subset of the manifold $\mathscr{C}^r(X, H')$ with the natural atlas.*

2. The manifold structure of $\mathscr{C}^r(X, Y)$. For a manifold X, we shall denote $\tau_X : T(X) \to X$ and $\tau_X^2 : T^2(X) \equiv T(T(X)) \to T(X)$ the (Banach) tangent bundles on X and $T(X)$ respectively, and $\Gamma(\pi)$ the set of all cross-sections of a (Banach) bundle $\pi : P \to X$, where P is the (Banach) bundle space. A *spray* on X is a cross-section $\xi \in \Gamma(\tau_X^2) \cap \Gamma(T\tau_X)$ such that for all $v \in T(X)$ and $\lambda \in R$, $\xi_{\lambda v} = \lambda Th_\lambda (\xi(v))$, where $h_\lambda : T(X) \to T(X)$ defined by the correspondence $v \longrightarrow \lambda v$. That is, a spray ξ on X is defined by the following commutative diagram:

$$\begin{array}{ccc} T^2(X) & \xrightarrow{Th_\lambda} & T^2(X) \\ \uparrow \xi & & \uparrow \xi \\ T(X) & \xrightarrow{h_\lambda} & T(X). \end{array}$$

For a spray ξ on X and a solution $\alpha_v(t)$ of ξ at a point $v \in T(X)$, with $t \in (-a, a)$ an open interval, let $\mathscr{D} \subset T(X)$ be the set of v in $T(X)$ such that $\alpha_v(1)$ is defined. The *exponential* of ξ is the mapping $exp^\xi : \mathscr{D} \to X$ defined by $v \longrightarrow \tau_X(\alpha_v(1))$ and we denote $Exp^\xi \equiv (\tau_X, exp^\xi) : \mathscr{D} \to X \times X$ by $v_x \longrightarrow (x, exp^\xi(x))$, where v_x is a point in the fibre over $x \in X$. Therefore the mapping Exp^ξ is of class C^r if X is a $\mathscr{C}^{r+2}$ manifold, $r \geq 1$.

Now let X be a compact C^r manifold, $r \geqq 1$, and Y a C^{r+s+2} manifold admitting partitions of unity. Let ξ be a C^{r+s} spray on Y. Then there exist a neighborhood $\mathscr{D}_\xi \subset T(Y)$ of the zero cross-section on Y and a neighborhood $\mathscr{F}_\xi \subset Y \times Y$ of the diagonal such that $Exp^\xi | \mathscr{D}_\xi : \mathscr{D}_\xi \to \mathscr{F}_\xi$ is a C^{r+s} diffeomorphism [1].

For each $f \in \mathscr{C}^r(X, Y)$, we can pull back the mapping Exp^ξ to the bundle space $f^*T(Y)$ induced from $T(Y)$ by f. Thus we have a diffeomorphism $\xi_f \equiv f^*Exp^\xi : f^*\mathscr{D}_\xi \to \mathscr{G}_{f,\xi}$, where $\mathscr{G}_{f,\xi} \subset X \times Y$ is a neighborhood of the graph (f). If we let $U_{f,\xi}$ be the set of all $g \in \mathscr{C}^r(X, Y)$ such that graph $(g) \subset \mathscr{F}_{f,\xi}$, the triple $(U_{f,\xi}, \varphi_{f,\xi}, \Gamma^r(f^*\tau_Y))$ is a chart of $\mathscr{C}^r(X, Y)$ at f, where $\varphi_{f,\xi}$: $U_{f,\xi} \to \Gamma^r(f^*\tau_Y)$ is defined by $g \longrightarrow \xi_f^{-1} \circ \mathrm{graph}(g)$ and we know that the set $\Gamma^r(f^*\tau_Y)$ of all C^r cross-sections of the induced bundle $f^*\tau_Y$ from the tangent bundle $\tau_Y : \mathrm{T}(Y) \to Y$ by f becomes a Banach space by the hypothesis that X is compact. The triple $(U_{f,\xi}, \varphi_{f,\xi}, \Gamma^r(f^*\tau_Y))$

is called a *natural chart* of $\mathscr{E}^r(X, Y)$ at f, and the maximal collection of all natural charts of $\mathscr{E}^r(X, Y)$ is called the *natural atlas* of $\mathscr{E}^r(X, Y)$. The following are a couple of the important results appeared in [1] as threorems:

(2.1) *If X is a compact C^r manifold, $r \geq 1$, and Y a C^{r+s+2} manifold admitting partitions of unity, the natural atlas of $\mathscr{E}^r(X, Y)$ is of class C^s, $0 \leq s \leq r$.*

(2.2) *With the same hypothesis as above, the evaluation mapping*

$$ev : \mathscr{E}^r(X, Y) \times X \longrightarrow Y$$

defined by the correspondence $(f, x) \longrightarrow f(x)$ is of class C^s, $0 \leqq s \leqq r$.

3. Proof of Theorm 1. By the asumption given to the manifold X, $\mathscr{I}^r(X, H')$ is a non-empty subset of $\mathscr{E}^r(X, H')$, where $H' = H \times R$, since so is $\varepsilon^r(X, H')$ due to (1.3). On the other hand, since H' is a separable Hilbert space that admits C^r partitions of unity, $\mathscr{E}^r(X, H')$ is a C^r manifold with the natural atlas due to (2.1).

We shall denote $L(E, F)$ the Banach space of all continuous linear mappings of a Banach space E into a Banach space F and $IL(E, F)$ the subset of $L(E, F)$ consisting of all splitting injections (i.e. injective mappings whose images split in F). The following is proved in [1].

LEMMA 1. *$IL(E, F)$ is a non-empty open subset of the Benach space $L(E, F)$.*

Due to (2.2) the evaluation mapping $ev : \mathscr{E}^r(X, H') \times X \longrightarrow H'$ is of class C^r, and if we consider a partial mapping with respect to the second factor $ev_f : \{f\} \times X \longrightarrow H'$, it is equivalent to the mapping $f : X \longrightarrow H'$ for each $f \in \mathscr{E}^r(X, H')$. Thus $D_2 ev : \mathscr{E}^r(X, H') \times X \longrightarrow L(H_0, H')$ is a C^{r-1} mapping such that $D_2 ev(f, x) = T_x f$, where H_0 is identified with $T_x(X)$ for all $x \in X$ and D_2 is the derivative operator with respect to the second factor. By Lemma 1, $\mathscr{U} = (D_2 ev)^{-1}(IL(H_0, H'))$ *is open in* $\mathscr{E}^r(X, H') \times X$. Suppose f is in $\mathscr{I}^r(X, H')$ it is obvious that $\{f\} \times X \subset \mathscr{U}$ by the usual criterion for immersions known as the Implicit Function Theorem on manifolds.

Since $\mathscr{U}$ is open, for each $x \in X$ we have an open subset $\mathscr{V}_x(f) \times U(f)$of $\mathscr{E}^r(X, H') \times X$ such that $\mathscr{V}_x(f) \times U(x)$ is contained in $\mathscr{U}$ with $v_x(f)$ and $U(x)$ being open neighborhood of f and x respectively. Let $\{U(x_i)\}^n_{i=1}$ be a finite open covering of X and $\mathscr{V}_{x_i}(f)$ the corresponding open sets in $C^r(X, H')$. If we let $\mathscr{V}(f) = \bigcap_{i=1}^n \mathscr{V}_{x_i}(f)$, $\mathscr{V}(f)$ is an open neighborhood of f in $\mathscr{E}^r(X, H')$ and moreover, for each $g \in \mathscr{V}(f)$ we have $\{g\} \times U(x_i) \subset \mathscr{U}$ for each $i = 1, \cdots, n$, since $\mathscr{V} x_i(f) \times U(x_i) \subset \mathscr{U}$ for each i. Taking the union for all i, $1 \leqq i \leqq n$,

$$\cup_i(\{g\} \times U(x_i)) = \{g\} \times \cup_i U(x_i) = \{g\} \times X \subset \mathscr{U}.$$

Therefore $\mathscr{V}(f) \times X \subset \mathscr{U}$. This implies that $\mathscr{V}(f) \subset \mathscr{I}^r(X, H')$, and hence

$\mathscr{G}^r(X, H')$ is open in $\mathscr{C}^r(X, H')$.

4. Proof of Theorem 2. If we denote $BL(E, F)$ the subset of the $L(E, F)$ consisting of all linear isomorphisms, where E and F are any Banach spaces as before, the following result is known [4]:

LEMMA 2. *$BL(E, F)$ is a non-empty open subset of $L(E, F)$.*

By its definition if $f \in \varepsilon^r(X, H')$, then there exists a submanifold Z of H' such that $f: X \to Z$ is a $\mathscr{C}^r$ diffeomorphism of X onto Z, and moreover $T_x f: H_0 \to H'$ is a linear isomorphism (into). Thus considering the evaluation mapping ev: $\mathscr{C}^r(X, H') \times X \to H'$ and its partial derivative $D_2 ev$ with respect to the second factor as before, it is not difficult to prove that $\varepsilon^r(X, H')$ is open in $\mathscr{C}^r(X, H')$ by the similar way done for the proof of Theorem 1 due to Lemma 2.

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