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Analytic Extensions and Local Spectra

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1. Introduction. The direct sum $H=\bigoplus_{\gamma\in\Gamma} H_\gamma$ of an arbitrary family of Hilbert spaces $\{H_\gamma\}_{\gamma\in\Gamma}$ is defined in the following: Let H be an additive subgroup $\prod_{\gamma\in\Gamma} H_\gamma$ consisting of all $(x_\gamma)\in\prod_{\gamma\in\Gamma} H_\gamma$ for which $\sum_{\gamma\in\Gamma}\|x_\gamma\|^2<\infty$.

Defining scalar multiplication and vector addition in H, coordinatewise and the scalar product in H by

$$(1,1)\qquad <(x_\gamma),\ (y_\gamma)>=\sum_{\gamma\in\Gamma}<x_\gamma,\ y_\gamma>,$$

Then $H=\bigoplus_{\gamma\in\Gamma} H_\gamma$ forms a Hilbert space.

Let $T_\gamma\in B(H_\gamma)$ be an endomorphism (that is a bounded linear operator on a complex Hilbert space to itself),

Suppose that

$$(1.2)\qquad sup\{\|T_\gamma\| : \gamma\in\Gamma\}<\infty.$$

We define the sum $T=\sum_{\gamma\in\Gamma} T_\gamma$ on H by

$$(1.3)\qquad T((x_\gamma))=(T_\gamma(x_\gamma)).$$

According to this definition, the operator T becomes an endomorphism, that is, $T\in B(H)$, and $\|T\|=sup\{\|T_\gamma\|:\gamma\in\Gamma\}$.

Defining a projection $P_\gamma : H\longrightarrow H_\gamma$ such that

$$(1.4)\qquad P_\gamma x=x_\gamma \text{ for each } x\in H,$$

We have the following.

(1.5)
(i) $\sum_{\gamma\in\Gamma} P_\gamma=I$
(ii) $P_\alpha P_\beta=\delta_{\alpha\beta}P_\alpha$, $(\alpha,\ \beta\in\Gamma)$
(iii) *The* restriction of T in H_γ is T_γ and
(iv) $TP_\gamma=P_\gamma T$

Obviously $P_\gamma x\in H_\gamma$, and together with the assertion (iv), We see that the operator T reduces each of the spaces H_γ, $\gamma\in\Gamma$.

Identifing an element x_γ in H_γ with the element $(0,\ 0,\cdots\cdots,\ 0,\ x_\gamma,\ 0,\ 0\cdots\cdots)$ in

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H, the space H_γ can be regarded as a subspace of the space H.

According to the above statement, it is easily seen that

$$Tx=\sum_{\gamma\in\Gamma} T_\gamma x_\gamma \text{ and } x=\sum_{\gamma\in\Gamma} x_\gamma.$$

The summations make senses since

$$\sum_{\gamma\in\Gamma}\|T_\gamma x_\gamma\|^2<\infty \text{ and } \sum_{\gamma\in\Gamma}\|x_\gamma\|^2<\infty.$$

In this paper we shall show the relation between the local spectrum of $T\in B(H)$ at $x\in H$ and the local spectra of T_γ at $x_\gamma\in H_\gamma$ for each $\gamma\in\Gamma$.

2. Notations, definitions and fundamental facts.

DEFINITION 1. *The local resolvent set $\rho(x,T)$ of T at x is the set of all complex numbers $\zeta\in C$ for which there is a neighborhood N of ζ, and an analytic function. $\mathscr{U}: N\to H$ such that $(\lambda I-T)\mathscr{U}=x$ for all $\lambda\in N$.*

The local spectrum $\sigma(x,T)$ of T at x is the complment in of $\rho(x,T)$ in C.

From this definition, We know that the analytic function $\mathscr{U}(\lambda, x)$ is an analytic continuation of the resolvent, i.e.,

$R(\lambda, x, T)=(\lambda I-T)^{-1}x$, $\lambda\in\rho(T)$, that is $\rho(T)\subset\rho(x, T)$ and

$\mathscr{U}(\lambda, x)=(\lambda I-T)^{-1}x$ if $\lambda\in\rho(T)$, and analytic function in the set $\rho(x, T)$

DEFINITION 2. *By an operator T having a single valued extension property, We mean that if there are two analytic extensions $\mathscr{U}_1$ and $\mathscr{U}_2$ of $(\lambda I-T)^{-1}x$, then $\mathscr{U}_1=\mathscr{U}_2$ on $N_1\cap N_2$. Where $N_i\cap\rho(x,T)\neq 0$ and $\mathscr{U}_i: N_i\to H (i=1, 2)$.*

It is a well known fact that an analytic extension need not be single valued, and that if T has the single valued extension property, then there is unique maximal extension whose domain is $\rho(x, T)$.

In the present papers we assume that the operator T. (or T_γ, $\gamma\in\Gamma$) has the single valued extension property.

3. Local Spectra. In this section, We will discuss the main purpose of this papers, first of all we have the following

LEMMA 1. *If an operator $T_\gamma: H_\gamma\to H_\gamma$ has the single valued extension property, then $\mathscr{U}(\lambda, x_\gamma)=P_\gamma\mathscr{U}(\lambda, x)$ for each $\gamma\in\Gamma$.*

Where $\mathscr{U}(\lambda, x_\gamma)$ and $\mathscr{U}(\lambda, x)$ are analytic extensions of $(\lambda I_\gamma-T_\gamma)^{-1}x_\gamma$ and $(\lambda I-T)^{-1}x$ respectively.

Proof. Since $P_\gamma T=TP_\gamma$ and $P_\gamma\in B(H)$, We have $\sigma(P_\gamma x, T)\subseteq\sigma(x, T)$ for all $x\in H$, therefore $\sigma(x_\gamma, T)\subseteq\sigma(x, T)$ for all $x_\gamma\in H_\gamma$ and $x\in H$.

By definition, for each $\zeta\in\rho(x, T)$, there exists a neighborhood $N(\zeta)$ and an analytic function $\lambda\to\mathscr{U}(\lambda, x)$ such that

(3.1) $$(\lambda I-T)\mathscr{U}(\lambda, x)=x \text{ for each } \lambda\in N(S).$$

Moreover we have

$$(\lambda I-T)\mathscr{U}(\lambda, x_\gamma)=x_\gamma \text{ since } \rho(P_\gamma x, T)\supseteq\rho(x, T).$$

An operator $\lambda I_\gamma-T_\gamma$ is a restriction of $\lambda I-T$ to H_γ and $\mathscr{U}(\lambda, x_\gamma)$ belongs to H_γ. Consequently we get

(3.2) $$(\lambda I_\gamma-T_\gamma)\mathscr{U}(\lambda, x_\gamma)=x_\gamma.$$

On the other hand

$$P_\gamma[(\lambda I-T)\mathscr{U}(\lambda, x)]=Px_\gamma=x_\gamma$$

Whence we have

(3.3) $$(\lambda I_\gamma-T_\gamma)P_\gamma\mathscr{U}(\lambda, x)=x_\gamma \text{ since } P_\gamma^2=P_\gamma.$$

From (3.2), (3.3) and the assumption that $T_\gamma(\gamma\in\Gamma)$ have single valued extension property, the equality

$$\mathscr{U}(\lambda, x_\gamma)=P_\gamma\mathscr{U}(\lambda, x)$$

must be satisfred.

LEMMA 2. *Suppose* $T\in B(H)$, $H=\bigoplus_{\gamma\in\Gamma}H_\gamma$, $P_\gamma H=H_\gamma$ *for each* $\gamma\in\Gamma$ *and* $x\in H$, *then we have*

(3.4) $$\sigma(x_\gamma, T)\subseteq\sigma(x_\gamma, T_\gamma) \text{ for each } \gamma\in\Gamma.$$

Proof. We know that the operator T is an extension of T_γ on H. Therefore the equation (3.2) implies

$$(\lambda I-T)\mathscr{U}(\lambda, x_\gamma)=x_\gamma \text{ for any } \lambda\in N(\zeta).$$

This shows $\zeta\in\rho(x_\gamma, T)$, that is, $\rho(x_\gamma, T_\gamma)\subseteq\rho(x_\gamma, T)$
thus we have (3.4).

THEOREM 1. *Suppose* $T\in B(H)$, $H=\bigoplus_{\gamma\in\Gamma}H_\gamma$, $P_\gamma H=H_\gamma$ *and* $T_\gamma\in B(H_\gamma)$. T *has the single valued extension property if and only if* T_γ *has the single valued extension property for each* $\gamma\in\Gamma$. *In this case, We have*

(3.5) $$\sigma(x, T)=\bigcup_{\gamma\in\Gamma}\sigma(x_\gamma, T_\gamma)$$

Proof. Suppose T_γ has the single valued extension property for each $\gamma\in\Gamma$. Then there exists unique extension of the resolvent of T_γ at x_γ such that

$$(\lambda I_\gamma-T_\gamma)\mathscr{U}(\lambda, x_\gamma)=x_\gamma \text{ for each } \lambda\in N(\zeta) \text{ if } \zeta\in\bigcap\rho(x_\gamma, T_\gamma).$$

Since $P_\gamma\mathscr{U}(\lambda, x)=\mathscr{U}(\lambda, x_\gamma)$ by Lemma 1, We have

$$P_\gamma[(\lambda I-T)\mathscr{U}(\lambda, x)-x]=0 \quad \text{for each } \gamma\in\Gamma.$$

Thus we have

$$(\lambda I-T)\mathscr{U}(\lambda, x)=x.$$

This means that $\mathscr{U}(\lambda, x)$ is an extended resolvent of T at x. It is easily seen that the H-valued analytic function $\mathscr{U}(\lambda, x)$ is unique, and that

$$\zeta\epsilon\rho(x, T), \text{ that is } \bigcap_{\gamma\epsilon\Gamma}\rho(x_\gamma, T_\gamma)\subset\rho(x, T).$$

Conversely, suppose that $T\epsilon B(H)$ has the single valued extension property, then there exist a unique extended resolvent such that

$$(\lambda I-T)\mathscr{U}(\lambda, x)=x$$

for each λ in some neighborhood of $\zeta\epsilon\rho(x, \mathrm{T})$.

Operating P_γ, We have

$$(\lambda I_\gamma-T_\gamma)\ P_\gamma\mathscr{U}(\lambda, x)=x_\gamma.$$

On the other hand, suppose that the extended resolvent of T_γ at x_γ is $\mathscr{U}(\lambda, x_\gamma)$, that is,

$$(\lambda I_\gamma-T_\gamma)\mathscr{U}(\lambda, x_\gamma)=x_\gamma.$$

Then $$(\lambda I_\gamma-T_\gamma)\ [P_\gamma\mathscr{U}(\lambda, x)-(\mathscr{U}(\lambda, x_\gamma)]=0.$$

Since λ is not an eigen value of T_γ, We have

$$P_\gamma\mathscr{U}(\lambda, x)=\mathscr{U}(\lambda, x_\gamma).$$

Therefore

$\zeta\epsilon\mathrm{p}(x_\gamma, T_\gamma)$ and hece we $P_\gamma\mathscr{U}(\lambda, x)$ is the unique extended resolvent of T_γ at x_γ for each $\gamma\epsilon\Gamma$. This shows $\mathrm{T}_\gamma(\gamma\epsilon\Gamma)$ has the sinle valued extension property at x_γ and $\rho(x, T)\subset\bigcup_{\gamma\epsilon\Gamma}\rho(x_\gamma, T_\gamma)$, thus we have completed the proof.

THEOREM 2. *If* $T\epsilon B(H)$, $H=\bigoplus_{\gamma\epsilon\Gamma}H_\gamma$ and $P_\gamma H=H_\gamma$ *for each* $\gamma\epsilon\Gamma$, *then*

$$\bigcup_{x\epsilon H}\sigma(P_\gamma x, T)=\sigma(T_\gamma). \tag{3.6}$$

Proof. For an arbitraly but fixed $\lambda\epsilon\rho(T_\gamma)$ and each $x_\gamma\epsilon H_\gamma$, there exists an analytic H_γ-valued function

$$f(\lambda)=(\lambda I-T_\gamma)^{-1}x_\gamma=(xI-T)^{-1}x_\gamma=(\lambda I-T)^{-1}P_\gamma x.$$

This shows that the number λ certainly belons to the resolvent set $\rho(P_\gamma x, T)$, thus we have

$$\rho(T_\gamma)\subseteq\rho(P_\gamma x, T) \text{ for each } x\epsilon H,$$

that is $$\bigcup_{x\epsilon H}\sigma(P_\gamma x, T)\subseteq\sigma(T_\gamma).$$

In order to have a converse relation, we consider an element $\zeta\epsilon\bigcap_{x\epsilon H}\rho(P_\gamma x, T)$.

By definition, there exists an analytic H_γ-valued function $\mathscr{U}(\lambda, P_\gamma x)$, and a neiborhood $N(\zeta)$ such that

$$(\lambda I-T)\mathscr{U}(\lambda, P_\gamma x)=P_\gamma x \text{ for each } x\epsilon H \text{ and } \lambda\epsilon N(\zeta).$$

Consider a mapping

$$A: x_\gamma\to\mathscr{U}(\lambda, x_\gamma).$$

Obviously, this mapping is linear on H_γ into H_γ. We consider the span of the set $\{\mathscr{U}(\lambda, x_\gamma): x_\gamma \in H_\gamma\}$ of analytic vector valued functions, and denote it by

$$S=\text{span }\{\mathscr{U}(\lambda, x_\gamma): x_\gamma \in H_\gamma\}.$$

Since a linear Combination of analytic fanctions is analytic for $\lambda \in \bigcap_{x\in H} \rho(P_\gamma x, T)$, the operator A maps H_γ onto S. Therefore

$$(\lambda I-T)\mathscr{U}(\lambda, x_\gamma)=(\lambda I-T)Ax_\gamma=x_\gamma.$$

Thus we have

$$\|x_\gamma\| \leqq \|\lambda I-T\|Ax_\gamma\| t \leq (|\lambda|+\|T\|)\|Ax_\gamma\|,$$

and

$$(|\lambda|+\|T\|)^{-1}\|x_\gamma\| \leq \|Ax_\gamma\|.$$

Since $\quad T \in B(H),\ M=(|\lambda|+\|T\|)^{-1}<\infty$ Whence

$$M\|x_\gamma\| \leq \|Ax_\gamma\|,\ M>0.$$

This shows that the operator A has a bounded inverse. The domain of A^{-1} is S and range of A^{-1} is all of H_γ, and the restriction of $\lambda I-T$ in S is the operator A^{-1}. Since A^{-1} is bounded and linear, this operator is Continuous. Therefore A^{-1} can be extended to a bounded operaton on $\bar{S}$. Hence $A=(A^{-1})^{-1}: H_\gamma \to \bar{S}$ is also closed, this means that $(\lambda I-T_\gamma)^{-1}$ is continuous, thus we have $\zeta \in \rho(T_\gamma)$.

Consequently,

$$\bigcap_{x\in H} \rho(P_\gamma x, T) \subseteq P(T_\gamma)$$

$$i.e., \ \sigma(T_\gamma) \subseteq \bigcup_{x\in H} \sigma(P_\gamma x, T).$$

It is well known that the equabity $\sigma(T_\gamma)=\bigcup_{x_\gamma \in H_\gamma} \sigma(x_\gamma, T_\gamma)$ is valid. From this together with Theorem 2, We have

$$\sigma(T_\gamma)=\bigcup_{x_\gamma \in H_\gamma} \sigma(x_\gamma, T_\gamma)=\bigcup_{x\in H} \sigma(P_\gamma x, T)$$

or

$$\bigcup_{x\in H} \sigma(P_\gamma x, T_\gamma)=\bigcup_{x\in H} \sigma(P_\gamma x, T).$$

Now we can prove the following.

THEOREM 3. *If* $H=\bigoplus_{\gamma\in\Gamma} H_\gamma$, $P_\gamma H=H_\gamma$ *and* $T \in B(T)$, *then following equatity will be satisfied*:

(3.7) $$\sigma(T)=\bigcup_{\gamma\in\Gamma} \sigma(T_\gamma)$$

where $$T=\sum_{\gamma\in\Gamma} T_\gamma \text{ and } T_\gamma \in B(H_\gamma).$$

Proof. By Theorem 1, $\bigcup_{\gamma\in\Gamma} \sigma(x_\gamma, T_\gamma)=\sigma(x, T)$, we have

$$\bigcup_{x\in H} \bigcup_{\gamma\in\Gamma} \sigma(P_\gamma x, T_\gamma)=\bigcup_{x\in H} \sigma(x, T)=\sigma(T)$$

And by the fact that $\bigcup_{x\in H} \sigma(P_\gamma x, T_\gamma)=\sigma(T_\gamma)$.

$$\bigcup_{\gamma\in\Gamma} \bigcup_{x\in H} \sigma(P_\gamma x, T_\gamma)=\bigcup_{\gamma\in\Gamma} \sigma(T_\gamma).$$

Therefore we have the desired equality

$$\sigma(T)=\bigcup_{\gamma\in\Gamma}\sigma(T_{\gamma}).$$

This completes the proof.

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