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On the Weak Primitivity of Matrix Rings

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1. Introduction. Let R be a ring. If there exists a faithful modular maximal right ideal I of R, R is called primitive [1]. Let R_n be the ring of $n\times n$ matrices, where n is a positive integer. E. C. Posner [5] proved that R_n is primitive if and only if R is primitive. The author defines a ring R to be weakly primitive if and only if there exists a faithful almost maximal right ideal I of R. The purpose of this note is to prove that R_n is weakly primitive if and only if R is weakly primitive, to mention that R_n is weakly transitive if and only if R is weakly transitive.

In section 2, some necessary fundamental comcepts including almost maximal right ideals, notations, and lemmas will be stated. In section 3, it will be proved that the weak primitivity of R implies that of R_n. In section 4, conversely it will be proved that the weak primitivity of R_n implies that of R. Finally it will be concluded that the weak primitivity of R_n is equivalent to that of R, with applic-ations this fact as corollaries.

2. Preliminaries. Let I be a right ideal of a ring R. Then I is called faithful if and only if R-I is a faithful (right) R-module, i.e.,

$$[I : R]=\{r\in R\mid Rr\subseteq I\}=\{0\}.$$

Evidently for $a\in R$,

$$[I : a]=\{r\in R\mid ar\in I\}$$

is a right ideal of R. It is to see that

$$N(I)=\{r\in R\mid\ rI\subseteq I\}$$

is the largest subring of R in which I is two-sided ideal. This subring $N(I)$ is called the normalizer of the right ideal I.

DEFINITION 1. A proper right ideal I of a ring R is called *almost maximal* if and only if

(1) for right ideals J_1, J_2, $J_1\cap J_2=I$ implies $J_1=I$ or $J_2=I$, that is, I is *meet irreducible*,

(2) $a\in R$ and $[I:a]\supset I$ implies $a\in I$,

(3) for a right ideal J such that $J\supset I$, $N(I)\cap J\supset I$ holds and also $a\in R$ such

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that $[J:a]\supseteq I$, $[J:a]\supset I$ holds.

A ring R is a right order in a ring L with a unity if and only if R is a subring of L, each regular (not zero divisor) element of L is a unit, and if $q\in L$ there exist a regular element $r\in R$ and an element $s\in R$ such that, in L, $q=sr^{-1}$.

Let M be a (right) R-module. Then M is called uniform if and only if every pair of non-zero submodules of M has a non-zero intersection. A weakly transitive ring of linear transformations is defined as follow:

DEFINITION 2. Let V be a left vector space over a division ring D and let R be a ring of linear transformations of V. Consider V as a (right) R-module. Then R is *weakly transitive* provided that there is a right order K in D and a (K,R)-submodule M of V such that M is uniform as a R-module, $DM=V$, and such that if $\{m_i\}_{i=1}^n$ is a finite D-linearly independent subset of M and if $\{y_i\}_{i=1}^n$ is a sequence from M, then there exists $r\in R$, $k\in K$, $k\neq 0$ such that $m_ir=ky_i$, $1\leqq i\leqq n$.

Useful lemmas will be stated whose prooves are to be found in [3] and [4] respectively.

LEMMA 1. *Let I be an almost maximal right ideal of a ring R. Then $M=R-I$ is a uniform (right) R-module with centralizer $K=N(I)/I$, and M has the extended centralizer D*[2] *in which K is a right order, and M can be extended to a (D,R)-module V such that $DM=V$ and such that R induces a weakly transitive ring of linear transformations of the vector space V over D.*

From the weak transitivity of this lemma, we draw a corollary which will be used later.

COROLLARY. *Let I be an almost maximal right ideal of a ring R. If $0\neq t\in I$, then there exists an element b in R such that $tb\in N(I)$ and $tb\notin I$.*

Proof. Since $t\notin$I, $\bar{t}\neq\bar{o}$ in $M=R-I$. For any non-zero $\bar{k}$ of $K=N(I)/I$, there exists a non-zero $\bar{h}$ in K and an element b in R such that $\bar{t}$b$=\bar{h}\bar{k}$, which is naturally not zero and belongs to $K=N(I)/I$, according to the weak transitivity of LEMMA 1.

Hence $tb\in N(I)$ but $tb\notin I$.

LEMMA 2. *If I is an almost maximal right ideal of a ring R and $M=R-I$, then for each $m\in M$, $m\neq 0$ the right ideal*

$$m^r=\{r\in R\mid mr=0\}$$

is almost maximal.

3. Weak Primitivity of R_n. As stated in Introduction, we proceed to show that the weak primitivity of R implies that of R_n.

PROPOSITION 1. *If there exists a faithful right ideal I of R, then there exists a faithful right ideal I^* of R_n.*

Proof. Suppose I is a right ideal of R such that $[I:R]=(0)$. Let

$$I^*=\{(r_{ij})\in R_n \mid r_{kj}\in I,\quad 1\leqq j\leqq n\}$$

for a fixed k. (Throughout this section k is a fixed positive integer.) Then I^* is a right ideal of R_n. If $(s_{ij})\in[I^*:R_n]$, then $R_n(s_{ij})\subseteq I^*$, that is, $rE_{kp}(s_{ij})\in I^*$ for all r of R, $1\leqq p\leqq n$. Therefore $rs_{pj}\in I$ for all r of R, $1\leqq j, p\leqq n$, namely, $Rs_{pj}\subseteq I$. Hence $s_{pj}=0$, $1\leqq j$, $p\leqq n$, because $[I:R]=(0)$. This means $(s_{ij})=(0)$ and proves that $[I^*:R_n]=(0)$, that is, I^* is faithful.

PROPOSITION 2. *Let I be an almost maximal right ideal of R, and let*

$$I^*=\{(r_{ij})\in R_n \mid r_{kj}\in I,\ 1\leqq j\leqq n\}.$$

If $J^*\supset I^*$ for a right ideal J^* of R_n, then there exist two matrices $\delta\in R_n$, $\beta\in J^*$ such that $\beta\delta\in N(I^*)\cap J^*$, $\beta\delta\notin I^*$, that is, $N(I^*)\cap J^*\supset I^*$, and $\beta\delta$ is the form in which every entity is 0 except the k-th column.

Proof. Let $\beta=(b_{ij})\in J^*, \beta\notin I^*$. Then there exists $b_{ks}\in R$ such that $b_{ks}\notin I$ for some s, $1\leqq s\leqq n$. From Corollary, it follows that there exists an element $d\in R$ such that $b_{ks}d\in N(I)$, $b_{ks}d\notin I$. Let $\delta=dE_{sk}$. Then $\beta\delta\in J^*$,

$$\beta\delta=\sum_i b_{is}dE_{ik}$$

and this matrix has the form in which every entity is 0 except the k-th column.

If $\alpha=(a_{ij})\in I^*$, then $(\beta\delta)\alpha\in I^*$ since the entities of the k-th row of $(\beta\delta)\alpha$:

$$b_{ks}da_{k1},\ b_{ks}da_{k2},\cdots,b_{ks}da_{kn}$$

are in I. Hence $\beta\delta\in N(I^*)$, and $\beta\delta\notin I^*$ since $b_{ks}d\notin I$.

PROPOSITION 3. *Let I and I^* be the same as in Proposition 2, and let J_s^*, $s=1,2$, be right ideals of R_n such that $J_s^*\supset I^*$. Then $J_1^*\cap J_2^*\supset I^*$.*

Proof. Let J_s, $s=1,2$, be the set of (k,k)-entities of all matrices which belong to J_s^*, $s=1,2$. Since it is clear that $I\subseteq J_s$, we show that $I\subset J_s$. Let $\beta_s\in J_s^*$, $\beta_s\notin I^*$. Then, by the argument in the proof of Proposition 2, there exists a matrix $\delta_s\in R_n$ for each s such that the (k,k)-entity of $\beta_s\delta_s$ belongs to J_s but not to I. Hence $J_s\supset I$, $s=1,2$. And consequently $J_1\cap J_2\supset I$ by the meet-irreducibility of I. Let $t\in J_1\cap J_2$, but $t\notin I$. Then we can choose an element $b\in R$ such that $tb\in N(I)$ but $tb\notin I$ by Corollary. On the other hand, there exist matrices $\beta_s\in J_s^*$, $s=1,2$, each of which is such that the (k,k)-entity is equal to t. And we can suppose, in particular, its every entity is 0 except the k-th row since $\beta_s=(t_{ij})\in J_s^*$ can be written as

$$\sum_{i,j}t_{ij}E_{ij}=\sum_j t_{kj}E_{kj}+\sum_{j\neq k}t_{ij}E_{ij}$$

where $\sum_{j\neq k}t_{ij}E_{ij}\in I^*\subset J_s^*$. Hence $\beta_1(bE_{kk})=\beta_2(bE_{kk})=tbE_{kk}\in J_1^*\cap J_2^*$, but $tbE_{kk}\notin I^*$. This proves that $J_1^*\cap J_2^*\supset I^*$.

PROPOSITION 4. *Let I and I^* be the same as in proposition 2. If $\beta=(b_{ij})\in R_n$ and $[I^*:\beta]\supset I^*$, then $\beta\in I^*$.*

Proof. First we show that $I\subseteq[I:b_{kj}]$, $1\leqq j\leqq n$. Let $a\in I$. Then $\beta aE_{j1}\in I^*$ since $aE_{j1}\in I^*$, $1\leqq j\leqq n$. Hence $b_{kj}a\in I$, *i,e.*, $I\subseteq[I:b_{kj}]$, $1\leqq j\leqq n$. From the hypothesis $[I^*:\beta]\supset I^*$, we can choose $\tau=(t_{ij})\notin I^*$ such that $\beta\tau\in I^*$. We assume that $t_{ks}\notin I$. If $j\neq k$, then $\beta t_{ks}E_{j1}\in I^*$ since $t_{ks}E_{j1}\in I^*$. Therefore $t_{ks}\in[I:b_{kj}]$. Hence $[I:b_{kj}]\supset I$. If $j=k$, $\beta(\sum_j t_{kj}E_{kj})\in I^*$, and $b_{kk}t_{ks}\in I$, since $\beta\tau\in I^*$,

$$\beta\tau=\beta(\sum_j t_{kj}E_{kj})+\beta(\sum_{i\neq j}t_{ij}E_{ij}),$$

where $\sum_{i\neq j}t_{ij}E_{ij}\in I^*$ and $\beta\sum_{i\neq j}t_{ij}E_{ij}\in I^*$. Therefore $t_{ks}\in[I:b_{kj}]$. Now it follows that $[I:b_{kj}]\supset I$, $1\leqq j\leqq n$. Hence $b_{kj}\in I$, $1\leqq j\leqq n$. This proves that $\beta\in I^*$.

PROPOSITION 5. *Let I and I^* be the same as in Proposition 2.*

If $J^*\supset I^*$ for a right ideal J^* of R_n and if $[J^*:\beta]\supseteq I^*$, then $[J^*:\beta]\supset I^*$.

Proof. Suppose $\beta=(b_{ij})\in R_n$ such that $\beta I^*\subseteq J^*$. Assume b_{kk} is in I and let $\gamma=\sum_j c_{kj}E_{kj}$, $\gamma\notin I^*$. Then $\beta\gamma\in I^*\subset J^*$, since every entity of the k-th row of $\beta\gamma$ is in I. Hence $[J^*:\beta]\supset I^*$. Now assume $b_{kk}\notin I$. And let J be the set of (k,k)-entities of all matrices in J^* whose entities are 0 except the k-th column. Then evidently J is a right ideal of R and $J\supseteq I$. This inclusion is proper, i.e., $J\supset I$, since there exists a matrix $(r_{ij})\in J^*\cap N(I^*)$, $(r_{ij})\notin I^*$ such that $r_{ij}=0$ for $j\neq k$, by Proposition 2. Also $b_{kk}I\subseteq J$, i.e., $[J:b_{kk}]\supseteq I$ since for every element $a\in I$, $aE_{kk}\in I^*$, $\beta aE_{kk}\in J^*$, and the (k,k)-entity $b_{kk}a$ of βaE_{kk} is in J. Hence $[J:b_{kk}]\supset I$, by the almost maximality of I. Now we can choose $c\in[J:b_{kk}]$, $c\notin I$. Let $\gamma=cE_{kk}$. Then $\beta\gamma\in J^*$, i.e., $\gamma\in[J^*:\beta]$ since $b_{kk}c$ is in J. But $\gamma\notin I^*$. Hence again $[J^*:\beta]\supset I^*$.

It is clear that from Proposition 1~Proposition 5, the following theorem holds.

THEOREM 1. Let R be a ring and let R_n be the ring of $n\times n$ matrices over R, where n is a positive integer. Then the weak primitivity of R implies that of R_n.

4. Weak Primitivity of R. Before we proceed to show that the weak primitivity of R_n implies that of R, we establish some properties.

LEMMA 3. *Let J^* be an almost maximal right ideal of R_n. Then there exists a matrix $\gamma=b_{pp}E_{pp}$ for some $b_{pp}\in R$ and for some p, $1\leqq p\leqq n$,* such that *$I^*=[J^*:\gamma]$ is an almost maximal right ideal of R_n, and $\gamma\in N(I^*)$, $\gamma\notin I^*$.*

Proof. Suppose $rE_{pq}\notin J^*$ for some $r\in R$ and for some p,q, $1\leqq p,q\leqq n$. Then there exists a matrix $\delta\in R_n$ such that $rE_{pq}\delta\in N(J^*)$, $rE_{pq\delta}\notin J^*$ by Corollary. Let $\beta=rE_{pq}\delta_1$ then β is a matrix whose every entity is 0 except the p-th row. Now let $\beta=(b_{ij})$ and $\gamma=b_{pp}E_{pp}$. Then $\gamma\beta=\beta^2$. Since $N(J^*)/J^*$ is an integral domain, $\beta^2\notin J^*,\gamma\beta\notin J^*$, thus $\gamma\notin J^*$. Evidently $I^*=\{\rho\in R_n\,|\,\gamma\rho\in J^*\}=[J^*:\gamma]$ is a right ideal of

R_n and its almost maximality follows from Lemma 2.

To show $\gamma \in N(I^*)$, suppose $\rho \in I^*$. Then $\gamma\rho \in J^*$ and, $\beta(\gamma\rho)$ is in J^* since $\beta \in N(J^*)$. But $\gamma^2 = \beta\gamma$, and $\beta\gamma\rho = \gamma^2\rho = \gamma(\gamma\rho)$. Hence $\gamma\rho \in I^*$, and $\gamma \in N(I^*)$.

Now suppose $\gamma \in I^*$. Then $\beta\gamma = \gamma\gamma \in J^*$, thus $\gamma \in [J^*:\beta]$. But $\gamma \notin J^*$ and $\beta \in N(J^*)$. Hence $[J^*:\beta] \supset J^*$ and $\beta \in J^*$, which is a contradiction. Therefore $\gamma \notin I^*$.

LEMMA 4. *Let $\gamma = b_{pp}E_{pp}$ and I^* be the same as in Lemma 3, and let I be the set of (p,p)-entities of all matrices in I^*. Then I is a right ideal of R and $b_{pp} \notin I$.*

Proof. It is clear that I is a right ideal of R. Now assume $b_{pp} \in I$. Then we can choose $\rho = (r_{ij}) \in I^*$ with $r_{pp} = b_{pp}$. Now $\rho\gamma \in I^*$ and

$$\rho\gamma = b_{pp}b_{pp}E_{pp} + \sum_{i \neq p} r_{ip}b_{pp}E_{ip}.$$

But $\sum_{i \neq p} r_{ip}b_{pp}E_{ip} \in I^*$ since $\gamma(\sum_{i \neq p} r_{ip}b_{pp}E_{ip}) = (0) \in J^*$. Hence $\gamma^2 = b_{pp}b_{pp}E_{pp} \in I^*$. Therefore $\gamma \in [I^*:\gamma]$. But $[I^*:\gamma] \supseteq I^*$ since $\gamma \in N(I^*)$. This is impossible because of Lemma 3.

PROPOSITION 6. *Let I be the same as in Lemma 4. Then, for a right ideal J of R such that $J \supset I$, $J \cap N(I) \supset I$ holds.*

Proof. Let γ and I^* be the same as in Lamma 4. Choose an element r such that $r \in J$, $r \notin I$ and define $\rho = rE_{pp}$. Then $\rho \notin I^*$. Therefore there exsts $\tau = (t_{ij}) \in R_n$ such that $\rho\tau \in N(I^*)$ but $\rho\tau \notin I^*$ by Corollary.

First, we show that $rt_{pp} \in N(I)$. Let $a \in I$. Then there exists $\sigma = (s_{ij}) \in I^*$ such that $s_{pp} = a$. Now $\gamma\sigma \in I^*$ and

$$\gamma\sigma = (b_{pp}E_{pp})(\sum_{i,j} s_{ij}E_{ij})$$
$$= \gamma(\sum_{l} S_{pl}E_{pl}).$$

Hence $\gamma(\sum_l s_{pl}E_{pl}) \in J^*$ and $\sigma_0 = \sum_l s_{pl}E_{pl} \in I^*$. Therefore, since $(\rho\tau)\sigma_0 \in I^*$ and

$$(\rho\tau)\sigma_0 = (rE_{pp})(\sum_{i,j} t_{ij}E_{ij})(\sum_l s_{pl}E_{pl})$$
$$= \sum_l rt_{pp}s_{pl}E_{pl},$$

$\sum_l rt_{pp}s_{pl}E_{pl} \in I^*$. Hence, in particular, $rt_{pp}s_{pp} = rt_{pp}a \in I$. It follows that $rt_{pp}I \subseteq I$, that is, $rt_{pp} \in N(I)$.

Next, to prove that $rt_{pp} \notin I$, suppose $\mu = (m_{ij}) \in I^*$ such that $m_{pp} = rt_{pp}$. *Now* $\mu\gamma \in I^*$ and

$$\mu\gamma = (\sum_{i,j} m_{ij}E_{ij})(b_{pp}E_{pp})$$
$$= m_{pp}b_{pp}E_{pp} + \sum_{i \neq p} m_{ip}b_{pp}E_{ip},$$

and $\sum_{i \neq p} m_{ip}b_{pp}E_{ip} \in I^*$. Therefore

$$m_{pp}b_{pp}E_{pp} = rt_{pp}b_{pp}E_{pp}$$

is in I^*. But

$$(\rho\tau)\gamma = (rE_{pp})(\sum_{i,j} t_{ij}E_{ij})(b_{pp}E_{pp})$$
$$= rt_{pp}b_{pp}E_{pp}.$$

Hence $(\rho\tau)\gamma \epsilon I^*$, that is, $\gamma \epsilon [I^* : \rho\tau]$. But $\gamma \notin I^*$ by Lemma 3. Therefore $[I^* : \rho\tau] \supset I^*$, which is a contradiction to the fact that $\rho\tau \notin I^*$. Hence $rt_{pp} \notin I$.

Now, we have proved that $J \cap N(I) \supset I$.

REMARK: In the proof of Proposition 6, we notice that if $r \notin I$ then there exists an element $t \epsilon R$ such that $rt \notin I$.

LEMMA 5. *Let I^* and I be the same as in Lemma 4. And let*

$$I^{**} = \{(r_{ij}) \epsilon R_n \mid r_{pj} \epsilon I, 1 \leqq j \leqq n\}.$$

Then $I^* = I^{**}$

Proof. To show that, $I^* \subseteq I^{**}$, suppose $\rho = (r_{ij}) \epsilon I^*$. Now $\rho(rE_{jp}) \epsilon I^*$, $1 \leqq j \leqq n$, and

$$\rho(rE_{jp}) = \sum_i r_{ij} r E_{ip}.$$

Hence $r_{pj} \epsilon I$ for all r of R, $1 \leqq j \leqq n$. By the above-mentioned Remark, $r_{pj} \epsilon I$, $1 \leqq j \leqq n$. This proves that $I^* \subseteq I^{**}$.

Conversely, suppose that $\rho = (r_{ij}) \epsilon I^{**}$, that is,

$$r_{p1}, r_{p2}, \cdots, r_{pn}$$

are in I, but $\rho = (r_{ij}) \notin I^*$. Then there exists $\tau \epsilon R_n$ such that $\rho\tau \epsilon N(I^*)$, $\rho\tau \notin I^*$ by Corollary. Let $\rho\tau = \sigma = (s_{ij})$. Then $s_{pp} \notin I$; for, if $\mu = (m_{ij}) \epsilon I^*$ with $m_{pp} = s_{pp}$, then

$$m_{pp} b_{pp} E_{pp} = s_{pp} b_{pp} E_{pp}$$

is in I^* where $\gamma = b_{pp} E_{pp}$, by the argument in the proof of Proposition 6, consequently $(\rho\tau)\gamma = \sigma\gamma \epsilon I^*$ by the argument in the proof of Lemma 4, leading to a contradiction as in the proof of Proposition 6. On the other hand, evidently every entity of the p-th row of $\rho\tau = \sigma$ is in I. This is again a contradiction. Therefore we conclude that $\rho = (r_{ij}) \epsilon I^*$, namely, $I^{**} \subseteq I^*$.

The above result $I^* = I^{**}$ will be used conviniently in the proof of the following proposition.

PROPOSITION 7. *Let I be the same as in Lemma 4. Then*

(1) *I is meet-irreducible.*

(2) *$[J:a] \supset I$ for a right ideal J such that $J \supset I$ and for an element a of R such that $[J:a] \supseteq I$.*

(3) *$a \epsilon R$ and $[I:a] \supset I$ implies $a \epsilon I$.*

Proof. (1) Assume J_1 and J_2 are right ideals of R and $J_1 \supset I$, $J_2 \supset I$. Let

$$J_s^* = = \{(r_{ij}) \epsilon R_n \mid r_{pj} \epsilon J_s, \ 1 \leqq j \leqq n\}, \ s = 1, 2.$$

Then J_1^* and J_2^* are right ideals of R_n. And $J_s^* \supset I^*$, since $I^* = I^{**}, s = 1, 2$. Hence $J_1^* \cap J_2^* \supset I^*$, from which it follows easily that $J_1 \cap J_2 \supset I$, since $I^* = I^{**}$.

(2) If $J \supset I$ and $a \epsilon R$ such that $aI \subseteq J$, let $\sigma = aE_{pp}$. And let

$$J^* = \{(r_{ij}) \epsilon R_n \mid r_{pj} \epsilon J, \ 1 \leqq j \leqq n\}.$$

Then $\sigma I^*\subseteq J^*$ since $I^*=I^{**}$. It follows that $[J^*:\sigma]\supset I^*$ by the almost maximality of I^*. Suppose $\tau=(t_{ij})\in[J^*:\sigma]$, but $\tau=(t_{ij})\notin I^*$, that is, $\tau=(t_{ij})\notin I^{**}$. Then the p-th row of $\sigma\tau$:

$$at_{p1}, at_{p2}, \cdots, at_{ps}, \cdots, at_{pn}$$

are in J, but $t_{ps}\notin I$ for some s, $1\leqq s\leqq n$. Hence $[J:a]\supset I$.

(3) Now suppose $[I:a]\supset I$ and $b\in[I:a]$, $b\notin I$. Let $\gamma=(c_{ij})\in I^*$, that is, $\gamma=(c_{ij})\in I^{**}$. Then

$$C_{p1}, C_{p2}, \cdots, C_{pn}$$

are in I, and

$$(aE_{pp})(\sum_{i,j}C_{ij}E_{ij})=\sum_j aC_{pj}E_{pj}$$

is in $I^*=I^{**}$ since

$$aC_{p1}, aC_{p2}, \cdots, aC_{pn}$$

are in I. Hence $aE_{pp}I^*\subseteq I^*$. On the other hand,

$$aE_{pp}bE_{pp}=abE_{pp}$$

is in $I^*=I^{**}$ since $ab\in I$ by assumption. Hence

$$bE_{pp}\in[I^*:aE_{pp}],$$
$$bE_{pp}\notin I^*$$

since $b\notin I$. Therefore $aE_{pp}\in I^*$. We have now $a\in I$.

Now that we have established the existence of an almost maximal right ideal I of R, it suffices to show that I is faithful in order to prove the following theorem.

THEOREM 2. *Let R and R_n be the same as in Theorem* 1. *Then the weak-primitivity of R_n implies that of R.*

Proof. Let J^* be the faithful almost maximal right ideal of R_n and let $M=R_n-J^*$ and $\alpha+J^*\in M$ for any $\alpha\in R_n$. Since $\gamma=b_{pp}E_{pp}\notin J^*$ where γ is the same as in Lemma 3, there exist $\rho=(r_{ij})\in R_n$ and $\sigma\in N(J^*)$ with $\sigma\notin J^*$, $\sigma+J^*\in N(J^*)/J^*$, such that

$$(b_{pp}E_{pp}+J^*)\rho=(\sigma+J^*)(\alpha+J^*)$$

by the weak transitivity in Lemma 1. Hence

$$\alpha+J^*=(\sigma+J^*)^{-1}(b_{pp}E_{pp}+J^*)\rho$$
$$=\delta(b_{pp}E_{pp}\rho+J^*),\quad \delta=(\sigma+J^*)^{-1}.$$

Let I and I^* be the same as in Lemma 4. If $s\in[I:R]$, then $Rs\subseteq I$, and, in particular, $r_{pp}s\in I$. Hence $r_{pp}sE_{pp}\in I^*$, since $I^*=I^{**}$. Now

$$(\alpha+J^*)(sE_{pp})=[\delta(\gamma\rho+J^*)](sE_{pp})$$

or

$$\alpha sE_{pp}+J^*=\delta(b_{pp}r_{pp}sE_{pp}+J^*).$$

Now since

$$b_{pp}r_{pp}sE_{pp}=\gamma(r_{pp}sE_{pp})$$

and $\gamma(r_{pp}sE_{pp})\in J^*$, it follows that $b_{pp}r_{pp}sE_{pp}\in J^*$, i.e., $b_{pp}r_{pp}sE_{pp}+J^*=\overline{0}$. Hence

$$\delta(b_{pp}r_{pp}sE_{pp}+J^*)=\overline{0}$$

or

$$\alpha sE_{pp}+J^*=\overline{0}.$$

Therefore $\alpha sE_{pp}\in J^*$, which means $sE_{pp}\in[J^*:R_n]$. But by the faithfulness of [illegible], $[J^*:R_n]=(0)$. Hence $s=0$ and consequently $[I:R]=(0)$. This proves I is faithful.

5. Conclusion. Now we can conclude as follow:

THEOREM 3. Let R be a ring and R_n be the ring of $n\times n$ matrices over R, where n is a positive integer. Then R_n is weakly primitive if and only if R is weakly primitive.

COROLLARY 1. *R_n is weakly transitive if and only if R is weakly transitive.*

Proof. This follows from the fact that a ring is weakly primitive if and only if it is weakly transitive [3].

COROLLARY 2. *If R_n(resp. R) is weakly primitive then R (resp. R_n) is prime.*

Proof. This follows the fact that a weakly primitive is prime [3].

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