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An Integral Estimate for an Elliptic System of Partial Differential Equations

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1. Introduction. Various estimates for elliptic systems of partial differential operators are studied by many authors [3]. In this paper, with the help of norm introduced in [5], we shall give an integral estimate in half spaces for an elliptic system with appropriate boundary conditions. We intend to discuss some applications of this result in future.

Before stating the theorem, we introduce some notations which will be used frequently in this paper. For a (real or complex) vector (x, t) where $x=(x_1, \cdots, x_{n-1})$ and t a scalar, its Euclidean norm is denoted by

$$|(x, t)|=(\sum_1^n x_j^2+t^2)^{\frac{1}{2}}$$

With a multi-index $\alpha=(\alpha_1, \cdots, \alpha_{n-1})$, we define

$$\partial_j=-i\frac{\partial}{\partial x_j}, \qquad \partial=(\partial_1, \cdots, \partial_{n-1}),$$

$$\partial^\alpha=\partial_1^{\alpha_1}\cdots\cdots\partial_{n-1}^{\alpha_{n-1}}, \qquad \delta=-i\frac{\partial}{\partial t},$$

where $|\alpha|=\alpha_1+\cdots+\alpha_{n-1}$ is called *the order of the differential operator* ∂^α.

The Fourier transform of $f=f(x, t)$ in C_c^∞ (a C^∞ function with compact support) will be denoted by $\hat{f}(\xi, \tau)$, where $\xi=(\xi_1, \cdots, \xi_{n-1})$ and τ a scalar.

For any function $K(\xi, \tau)$, we define the symbolic operator $K(\partial, \delta)$ by the formula

$$K(\partial, \delta)f(x, t)=\frac{1}{(2\pi)^{\frac{n}{2}}}\iint e^{i(x\xi+t\tau)}K(\xi, \tau)\hat{f}(\xi, \tau)d\xi d\tau.$$

i.e. $K(\partial, \delta)f(x, t)$ is the function whose Fourier transform is $K(\xi, \tau)\hat{f}(\xi, \tau)$ provided that $K(\xi, \tau)\hat{f}(\xi, \tau)\in L^2(E_n)$.

The partial Fourier transform $\tilde{f}(\xi, t)$and the operator $K(\partial)$ with respect to the first $n-1$ variables $x=(x_1, \cdots, x_{n-1})$ are defined analogously.

For any real number s and p, we set

$$(1.1)\qquad \|f\|_{s,p}=\frac{1}{(2\pi)^{n/2}}\iint|(1+|\xi|)^p\{\tau+i(|\xi|+1)^s\}\hat{f}(\xi, \tau)|^2 d\xi d\tau$$
$$=\|(1+|\partial|)^p\{\delta+i(|\partial|+1)^s\}f\|_{L^2(E_n)}$$

By $H^{s,p}$, we denote the completion of C_c^∞ with respect to the norm $\|\cdot\|_{s,p}$.

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$H^{s,p}_{-}$ is the subspace of elements in $H^{s,p}$ whose support is contained in the half space $t\leqq 0$. $(H^{+})^{s,p}$ is the quotient space $H^{s,p}/H^{s,p}_{-}$ and $\|\cdot\|^{+}_{s,p}$ the quotient norm.

For $f(x, t)$ in $(H^{+})^{s,p}$, let $\mathring{f}(x, t)$ be an element in $H^{s,p}$ whose restriction to $t\geqq 0$ is f. Let $\tilde{f}(\xi, t)$ be the partial Fourier transform of f. Then we have the following important formula which follows from the definition of $\|\cdot\|^{+}_{s,p}$,

(1.2) $$\|f\|^{+}_{s,p}=\iint_{t>0}|(|\xi|+1)^{p}(\delta+i|\xi|+1)^{s}\tilde{f}(\xi, t)|^{2}d\xi dt.$$

Finally $\|\cdot\|'_{p}$ and $(H')^{p}$ denotes usual $\|\cdot\|_{p}$ and H^{p} for functions depending on x only. Thus

(1.3) $$\|f\|'_{p}=\|(|\partial|+1)^{p}f\|'_{L^2}$$

for C_c^{∞} functions f which depend only upon x and $(H')^{p}$ is the completion of C_c^{∞} with respect to $\|\cdot\|'_{p}$.

2. Statement of the problem. Let $l_{i,j}(\partial, \delta)$; $i, j=1, \cdots, N$, be a system of partial differential operators of the form

(2.1) $$l_{i,j}(\partial, \delta)=\sum_{|\alpha|+\beta\leqq s_i+t_j} a^{i,j}_{\alpha,\beta}\partial^{\alpha}\partial^{\beta}$$

where $a^{i,j}_{\alpha,\beta}$ are constants and s_i an index <0. The sum of the terms in $l_{i,j}(\partial, \delta)$ with the order s_i+t_j are denoted by $l'_{i,j}$.

Condition I ;

(i) $det(l'_{i,j}(\xi, \tau))\neq 0$ for any real non-zero vector (ξ, τ). $det(l'_{i,j}(\xi, \tau))$ as a polynomial in (ξ, τ) is of even degree 2m.

(ii) For any non-zero real vector ξ, $det(l'_{i,j}(\xi, \tau))$ in complex variable τ has exactly m zeros with positive imaginary parts.

Let the zeros with positive imaginary parts (for given ξ) be $\tau_i^{+}(\xi)$; $i=1, \cdots, m$. We set

(2.2) $$L^{+}(\xi, \tau)=\prod_{i=1}^{m}(\tau-\tau^{+}_{i}(\xi)).$$

We further denote the cofactor of $l'_{i,j}$ in $det\ (l'_{i,j})$ by $L^{i,j}$.

Next we consider a system of boundary operators

(2.3) $$\gamma_{h,j}(\partial, \delta)=\sum_{|\alpha|+\beta\leqq q_h+t_j} b^{h,j}_{\alpha,\beta}\partial^{\alpha}\delta^{\beta} \qquad \begin{array}{l} h=1, \cdots, m \\ j=1, \cdots, N\end{array}$$

where $b^{h,j}_{\alpha,\beta}$ are constants and $\gamma_{h,j}=0$ when $q_h+t_j<0$. $\gamma'_{h,k}$ is defined to be the sum of the terms in $\gamma_{h,k}$ which are of order q_k+t_j. For the boundary system $\gamma_{k,k}$ we make the following assumption called complementing condition:

Condition II **:**

(2.4) $$\sum_{h=1}^{m} c_h\sum_{j=1}^{N}\gamma'_{h,j}L^{j,k}\equiv 0 \qquad (\mathrm{mod}\ L^{+})$$

if and only if the constants $c_h=0$, $h=1, \cdots, m$.

Under these assumptions, we shall prove the following inequality:

THEOREM *If* $(l_{i,j})$ *and* $(\gamma_{h,j})$ *satisfy the conditions* I, II, *then the following inequality holds for all* $u=(u_1, \cdots, u_N)$ *in* $(H^+)^{s+t_1,p}\times\cdots\times(H^+)^{s+t_N,p}$

$$(2.5)\qquad \sum_{j=1}^{N}\|u_j\|_{s+t_j}\leqq c\Big(\sum_{i=1}^{N}\|l_{i,j}\,u_j\|^{+}_{s-s_i,p}$$

$$+\sum_{h=1}^{m}\|\gamma_{h,j}\,u_j\|'_{s-q_h-\frac{1}{2}+p}+\sum_{j=1}^{N}\|u\|_{s+t_j,p-1}\Big),$$

where s *is a real* $\geqq max\left(0,\ q_h+\dfrac{1}{2}\right)$ *and* c *is a constant independent of* u.

We note that the summation convention is employed in the expression (2.5).

3. Preliminary remarks. Let us first consider the problem for the ordinary differential equations

$$(3\cdot1)\qquad \begin{aligned}&\lambda_{i,j}(\delta)u_j(t)=f_i(t) && (t>0),\ i=1,\cdots,N,\\ &\rho_{h,j}(\delta)u_j(t)=\varphi_h && (t=0),\ h=1,\cdots,m,\end{aligned}$$

which satisfy the following conditions which correspond to I, II of the previous section.

Condition I′;

det $(\lambda_{i,j}(\tau))$ is of order $2m$ in τ, has no real zeros, and the equation *det*$(\lambda_{i,j}(\tau))=0$ has exactly m zeros τ^+_i, $i=1,\cdots,m$ with positive imaginary parts.

As before, we denote the cofactor of $\lambda_{i,j}$ in *det*$(\lambda_{i,j})$ by $\Lambda^{i,j}$. We also set

$$\Lambda^+=\prod_{i=1}^{m}(\tau-\tau^+_i).$$

Condition II′

$$(3.2)\qquad \sum_{h=1}^{m}c_h\sum_{j=1}^{N}\lambda_{h,j}\,\Lambda^{j,k}\equiv0\ (\Lambda^+)$$

if and only if the constants $c_h=0$, $h=1,\cdots,m$.

Under these assumptions it is proved in [3] that the problem (3.1) has a unique exponentially decaying solution for each exponentially decaying $f=(f_1,\cdots,f_N)$ and complex $\varphi={}^t(\varphi_1,\cdots,\varphi_m)$. Hence we see from the definition of the spaces concerned that the linear mapping

$$(u_1,\cdots,u_N)\to((l_{1,j}u_j,\cdots,l_{N,j}u_j:\gamma_{1,j}u_j(0),\cdots,\gamma_{m,j}u_j(0))$$
$$(H^+)^{s+t_1}\times\cdots\times(H^+)^{s+t_N}\to(H^+)^{s-s_1}\times\cdots\times(H^+)^{s-s_N}\times C\times\cdots\times C$$

is $1-1$ and continuous. Here the spaces $(H^+)^s$ are 1-dimensional analog of $(H^+)^{s,0}$ for n-dimensional case. Therefore by closed graph theorem we have the inequality

$$(3.3)\qquad \sum_{j=1}^{N}\|u_j\|_{s+t_j}\leqq c\Big(\sum_{i=1}^{N}\|l_{i,j}\,u_j\|_{s-s_i}+\sum_{h=1}^{m}|\gamma_{h,j}\,u_j(0)|\Big).$$

4. Proof of the theorem. In proving the theorem, it is sufficient to assume that $l_{i,j}=l'_{i,j}$, $\gamma_{h,j}=\gamma'_{h,j}$, i.e. the operators are homogeneous with the highest order terms only. Proof for the general case can be carried out by routine method (*c.f.* [1]).

Let us apply (3.3) to the system

$$l_{i,j}(\xi,\delta)u_j(\xi,t)=f(\xi,t),\quad t>0 \tag{4.1}$$
$$\gamma_{h,j}(\xi,\delta)u_j(\xi,t)=\varphi_j(\xi,t),\quad t>0$$

with $|\xi|=1$. Then we obtain

$$\sum_j\int_{t>0}(\delta+i)^{s+t_j}u_j(\xi,t)|^2dt \tag{4.2}$$
$$\leqq c^2\Big[\sum_i\int_{t>0}|(\delta+i)^{s-s_i}l_{i,j}(\xi,\delta)u_j(\xi,t)|^2dt$$
$$+\sum_h|\gamma_{h,j}(\xi,\delta)u_j(\xi,0)|^2\Big],\quad |\xi|=1$$

where c is a constant independent of ξ.

In (4.2), let us replace $l_{i,j}(\xi,\delta)$, $\gamma_{h,j}(\xi,\delta)$ and $u_j(\xi,t)$ by $l_{i,j}(\xi/|\xi|,\delta)$, $\gamma_{h,j}(\xi/|\xi|,\delta)$ and $u_j(\xi,t/|\xi|)$ where $|\xi|\geqq1$. Then we have

$$\sum_j\int_{t>0}|(\delta+i|\xi|)^{s+t_j}u_j(\xi,t)|^2dt \tag{4.3}$$
$$\leqq c^2\Big[\sum\int_{t>0}|(\delta+i|\xi|)^{s-s_i}l_{i,j}(\xi,\delta)u_j(\xi,t)|^2dt$$
$$+\sum_h|\xi|^{2\left(s-q_h-\frac{1}{2}\right)}|\gamma_{h,j}(\xi,\delta)u_j(\xi,0)|^2\Big],\quad |\xi|\geqq1.$$

Since the norms $\left(\int_{t>0}|(\delta+i|\xi|)^s u(\xi,t)|^2dt\right)^{\frac{1}{2}}$ and $\left(\int_{t>0}|(\delta+i(|\xi|+1))^s u(\xi,t)|^2dt\right)^{\frac{1}{2}}$ are uniformly equivalent (with respect to ξ), we have

$$\sum_j\int_{t>0}|(\delta+i(|\xi|+1))^{s+t_j}u_j(\xi,t)|^2dt \tag{4.4}$$
$$\leqq c^2\Big[\sum_i\int_{t>0}|(\delta+i(|\xi|+1))^{s-s_i}l_{i,j}(\xi,\delta)u_j(\xi,t)|^2dt$$
$$+\sum_h(1+|\xi|)^{2\left(s-q_h-\frac{1}{2}\right)}|\gamma_{h,j}(\xi,\delta)u_j(\xi,0)|^2\Big]$$

with different constant c.

Multiplying both sides of (4.4) by $(1+|\xi|)^{2p}$ and integrating with respect to ξ over the set $|\xi|\geqq1$, we have

$$\sum_j\int_{t>0}dt\int_{|\xi|\geqq1}|(\delta+i(1+|\xi|))^{s+t_j}(1+|\xi|)^p u_j(\xi,t)|^2d\xi \tag{4.5}$$
$$\leqq c^2\Big[\sum_i\int_{t>0}dt\int_{|\xi|\geqq1}|(1+|\xi|)^p(\delta+i(1+|\xi|))^{s-s_i}l_{i,j}(\xi,\delta)u_j(\xi,t)|^2d\xi$$
$$+\sum\int_{|\xi|\geqq1}|(1+|\xi|)^{s-q_h-\frac{1}{2}+p}\gamma_{h,j}(\xi,\delta)u_j(\xi,0)|^2d\xi\Big]$$

$$\leqq c^2\Big[\sum_i(\|l_{i,j}(\partial,\delta)u_j\|^{+}_{s-s_i,p})^2+\sum_h(\|\gamma_{h,j}(\partial,\delta)u_j\|'_{s-q_h-\frac{1}{2}+p}\Big]$$

on the other hand

$$\int_{t>0}dt\int_{|\xi|\geqq 1}|(1+|\xi|)^p(\delta+i(|\xi|+1))^{s+t_j}u_j(\xi,t)|^2d\xi\leqq 4(\|u_j\|_{s+t_j,p-1}) \tag{4.6}$$

Then adding (4.5) and (4.6), we obtain the desired inequality (2.5.).

References

1, Agmon, S., *Lecture note* 1964.

2, Agmon, S., Douglis, A., Nirenberg, L., *Estimates near the boundary for solutions of elliptic partial differential equations* I, C.P.A.M., Vol. 12, 1959, pp 623–727.

3. Agmon, S., Douglis, A., Nirenberg, L., *Estimates near the boundary for solutions of elliptic partial differential equations* II, C.P.A.M. Vol 17, 1964, pp 35—92.

4. Douglis, A., *Dirichlet's problem for linear elliptic partial differential equations of second and higher order*, *Lecture note*, Univ. of Maryland, 1958.

5. Peetre, J., *Another approach to elliptic boundary value problems*, C.P.A.M. Vol 14, 1961, pp 711—731.

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