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《Original》

Green's Function of Time-Energy Dependent Neutron Transport Equation

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Abstract

The spectrum of continuous transfer operator arising in a time-energy dependent neutron transport equation is analyzed. Four theorems concerning on the spectrum are proved. A convolution theorem of the generalized Mellin energy transform is given. Also the completeness theorem necessary for a final solution is proved. A unique time decay constant 1 - c is found, which is dominant asymptotically.

요 약

시간과 에너지에 종속된 중성자 전도 방정식에 나타나는 연속 에너지 전도 연산자의 스펙트럼 (Spectrum)을 분석했다.

스펙트럼에 관한 4가지 정리를 증명하고 일반화된 Mellin 에너지변화의 Convolution 정리를 얻었다. 또한 최종해에 필요한 완전성정리를 증명하고 접근적으로 가장 우세한 시간붕괴상수 1 - c를 발견하였다.

1. Introduction

Most theoretical investigations on the eigenvalue spectrum of the neutron transport equation originate from Lehner and Wing's [1,2] fundamental papers, where the time-dependent one-velocity transport equation was first fully and rigorously treated. These authors showed that the eigenvalue spectrum of this equation (considered in a Hilbert space of square summable function) is made of the following parts: a *continuous* sprectrum and a point spectrum consisting of a finite (but not zero) number of real eigenvalues.

In this paper we will treat more general problem, i. e., a time-energy dependent neutron transport equation with a continuous energy transfer operator. An importance of a continuous energy transfer operator is first pointed out by a pulsed neutron transport problem by Corngold[3]. A theory of neutron transport equation is an application of a singular integral equation[4,5] and a spectrum theory of integral transform operator[6,7].

We will depend heavily upon the result of both these two fields.

2. Spectrum analysis

A theory of a plane symmetric, pulsed energy dependent neutron transport equation, with an isotropic source at $x=0$, after a Fourier transform in x-coordinate, is described by the following linearized Boltzmann equation[8].

$$\left[\frac{\partial}{\partial t}+ik\mu+1\right]\Psi_k(\mu,E,t)=$$
$$\frac{c_F}{2}\int_{-1}^{1}d\mu'\chi(E)\int_0^\infty\Psi_k(\mu',E',t)\,dE'$$
$$+\frac{c_i}{2}\int_{-1}^{1}d\mu'\int_0^\infty dE'K_{in}(E'\to E)\Psi_k(\mu',E',t)$$
$$+\frac{1}{2}\delta(t)S(E), \tag{2.1}$$

where $\mu=\cos\theta$, k, c_i, c_F are some parameters, $\chi(E)$ is a known function of E and $K_{in}(E'\rightarrow E)$ is the energy transfer operator, defined as below, but neither self-adjoint nor symmetric, $\Psi_k(\mu,E,t)$ are unknown function to be determined.

The kernel $K_{in}(E'\rightarrow E)$ is defined as follows:

$$K_{in}(E'\rightarrow E)=f(E')g(E), \quad \text{if } E'\geq E, \qquad (2.2)$$
$$=0, \quad \text{if } E'<E,$$

and

$$\int_0^E K_{in}(E'\rightarrow E)dE=1. \qquad (2.3)$$

From the Eqs. (2.2) and (2.3), we have, defining

$$h(E)=1/f(E), \qquad (2.4)$$

the important relationship

$$g(E)=\frac{d}{dE}h(E). \qquad (2.5)$$

From a physical model, an energy dependence of the function $g(E)$ is given explicitly as

$$g(E)=E/T^2e^{-E/T}, \qquad (2.6)$$

where T=const, therefore integrating with respect to E, we get

$$h(E)=1-(1+E/T)e^{-E/T}. \qquad (2.7)$$

We note that $g(E)=0$ for $E=0$ and $g(E)\rightarrow 0$ for $E\rightarrow\infty$ and $h(E)=0$ for $E=0$, $h(E)\rightarrow 1$ for $E\rightarrow\infty$.

Next we will study a spectral properties of this kernel $K_{in}(E'\rightarrow E)$ with the explicitly given form of $g(E)$ and $h(E)$ in Eqs. (2.6) and (2.7). So it is quite useful to define an energy operator O_E as

$$O_E\Psi_k(\mu,E,t)=g(E)\int_E^\infty\frac{\Psi_k(\mu,E',t)}{h(E')}dE'. \qquad (2.8)$$

Notice that from the explicit form of $g(E)$, $g(E)$ is everywhere positive, and so is $h(E)=\int_0^E g(E')dE'$.

It is important to notice that for some fixed positive number $\varepsilon>0$

$$g(\varepsilon)\neq 0, \quad h(\varepsilon)\neq 0$$

and $h(\infty)=\int_0^\infty g(E')\,dE'<\infty \qquad (2.9)$

We use the functional energy-space $C[\varepsilon,\infty]$, i.e., the Banach space of continuous functions on $[\varepsilon,\infty]$.

Let $\phi(E)\in L_1[\in,\infty]$.

In this space, the norm is defined as

$$||\phi||=\int_\varepsilon^\infty|\phi(E')|dE'. \qquad (2.10)$$

Theorem 2.1. As an operator acting in the space L_1, O_E is linear and *continuous.*

Proof: A linearity of O_E is obvious.

Let $\Phi(E)=O_E\phi(E)=g(E)\int_E^\infty\frac{\phi(E')}{h(E')}dE'.$

Then

$$||\Phi||=\int_\varepsilon^\infty |\Phi(E')|dE'$$
$$||\Phi||\leq\int_\varepsilon^\infty dE|g(E)|\int_E^\infty\frac{|\phi(E')|}{|h(E')|}dE'.$$

We may invert the order of integration. Thus

$$||\Phi||\leq\int_\varepsilon^\infty dE'\frac{|\phi(E')|}{|h(E')|}\int_\varepsilon^{E'}|g(E)|dE$$
$$||\Phi||\leq\int_\varepsilon^\infty\frac{|\phi(E')|}{h(E')}[h(E')-h(\varepsilon)]dE'$$

Since $|g(E)|\equiv g(E)$, $|h(E)|\equiv h(E)$,

$$||\Phi||\leq\frac{\sup h(E)-h(\varepsilon)}{\inf h(E)}||\phi||$$
$$||\Phi||\leq M||\phi||,$$

since $\inf h(E)\neq 0$ for $E\in[\varepsilon,\infty]$ and where M is some constant. Q.E.D.

Theorem 2.2. In $C[\varepsilon,\infty]$, O_E is a compact operator.

Proof: Defining $K_{in}(E'\rightarrow E)$ through Eqs. (2.2), (2.3), (2.4), (2.6) and (2.7), and by physical reason, E' and E have the meaning of incoming and outgoing neutron energy respectively, therefore it is true that $E<E'<E_o$ always. So we have

$$\iint_\triangle dE\,dE'|K_{in}(E'\rightarrow E)|^2<\infty$$

in the domain $\triangle$ defined by $\varepsilon<E'<E_o$ and $\varepsilon<E<E_o$. From Taylor's theorem[7] (p. 276), this is enough to assure the compactness of O_E in $C[\varepsilon, E_o]$

Theorem 2.3. The point spectrum of O_E contains, at most, a countable set of points, and these have no accumulation point, except possibly at ∞.

Proof: We quote Taylor's theorem[7] (p.281); "Suppose $T\in[X]$ and T compact. Then $P_\sigma(T)$ contains at most a countable set of points and these have no accumulation point, except possible at $\lambda=\infty$". Recall the definition of the spectrum being as the *complement* of the set $\{\lambda\}$ such that: $(I-\lambda O_E)^{-1}$ exists as an operator and is *continuous.*

Theorem 2.3 does not imply that the point spectrum is not empty. Let us look specifically for eigenfunctions of O_E; if λ belongs to the point spectrum

$$(1-\lambda O_E)\phi_\lambda(E)=0 \qquad (2.11)$$

The $\phi_\lambda(E)$ being the associated eigenfunction ——Eq. (2.11) can be rewritten as:

$$\phi_\lambda(E)-\lambda g(E)\int_\varepsilon^{E_0}\frac{\phi_\lambda(E')}{h(E')}dE'=0 \text{ for } E>\varepsilon. \quad (2.12)$$

Put $\Psi(E)=\phi_\lambda(E)/g(E)$. (2.12a)

And Eq. (2.12) becomes

$$\Psi(E)-\lambda\int_E^{E_0}\frac{g(E')}{h(E')}\Psi(E')dE'=0 \quad (2.13)$$

We differentiate (2.13) and obtain

$$\frac{d\Psi}{dE}=-\lambda\frac{g(E)}{h(E)}\Psi(E) \quad (2.14)$$

Keeping in mind Eq. $g(E)=-\frac{d}{dE}h(E)$, the solution of Eq. (2.14) is straightforward.

$$\Psi(E)=K[h(E)]^{-\lambda},$$

where K is arbitrary, and from Eq. (2–12a)

$$\phi_\lambda(E)=Kg(E)[h(E)]^{-\lambda}. \quad (2.15)$$

But Eq. (2.15) is not equivalent to the initial Eq. (2.12) and we must verify that $\phi_\lambda(E)=g(E)\times h(E)^{-\lambda}$ is indeed a solution of the initial equation.

$$\phi_\lambda(E)-\lambda g(E)\int_E^{E_0}\frac{\phi_\lambda(E')}{h(E')}dE'$$

$$=g(E)h(E)^{-\lambda}-\lambda g(E)\int_E^{E_0}g(E')h(E')^{-\lambda-1}dE'$$

$$=g(E)h(E)^{-\lambda}+g(E)\Big[h(E)^{-\lambda}\Big]_E^{E_0}=g(E)h(E_0)^{-\lambda}\neq 0,$$

Since $h(E_0)\neq 0$ even for $E_0\to\infty$, Eq. (2.16) expresses that $g(E)\ h(E)^{-\lambda}$ is *not* a solution of Eq. (2.12).

This means that the point spectrum is empty. Indeed, the result is stronger.

Theorem 2.4. The whole spectrum (continuous, point, residual) of O_E is empty, except for the point at ∞.

Proof: Let us show that the operator $(I-\lambda O_E)^{-1}$ exists for any λ, $\lambda\neq\infty$. Given $S(E)$ arbitrary, $S(E)\in C[\varepsilon,E_0]$, the existence of $(I-\lambda O_E)^{-1}$ is equivalent to the existence of a solution $\phi(E)\in C[\varepsilon,E_0]$ to the equation.

$$(I-\lambda O_E)\phi(E)=S(E). \quad (2.17)$$

$$i.e.\ \phi(E)-\lambda g(E)\int_E^{E_0}\frac{\phi(E')}{h(E')}dE'=S(E). \quad (2.18)$$

From the result (2.16), we find easily that the solution of more specialized equation

$$\phi(E)-\lambda g(E)\int_E^{E_0}\frac{\phi(E')}{h(E')}dE'=\delta(E-E_0) \quad (2.19)$$

is

$$\phi(E)=\delta(E-E_0)+\frac{\lambda g(E)h(E)^{-\lambda}}{h(E_0)^{1-\lambda}}. \quad (2.20)$$

From Eqs. (2.19) and (2.20) we find the final solution of (2.18)

$$\phi(E)=S(E)+\lambda g(E)h(E)^{-\lambda}\int_E^{E_0}\frac{S(E')}{h(E')^{1-\lambda}}dE'. \quad (2.21)$$

This means that operator $(1-\lambda O_E)^{-1}$ exists. Then, since O_E is compact (Theorem 2.2), $(1-\lambda O_E)^{-1}$ is *continuous* (as proved in Taylor[7], p.281 "Suppose $T\in[X]$, T compact, and $\lambda\neq 0$. Then $(\lambda-T)^{-1}$ is continuous if it exists."). So $\lambda\neq\infty$ belongs to the resolvent set of O_E. Since the spectrum of any linear continuous operator is not empty, (Taylor[7], p. 261: "if $T\in[X]$ and X is a complex Banach space, $\sigma(T)$ is not empty."), it means that, the spectrum O_E is reduced to $\lambda=\infty$. Q.E.D.

Theorem 2.4 is also valid even if we extend E_0 to ∞.

3. Generalized Mellin Energy Transform

We have found *pseudo*-eigenfunction of O_E namely:

$$\phi_\lambda(E)=g(E)\ h(E)^{-\lambda}.$$

This will be used for a new energy transformation of the original kernel of integral equation (2.1) in order to reduce it to a known standard form of time dependent monokinetic equation, whose solution is well-known. Let us introduce the following fundamental energy transform:

$$\bar{\Psi}_k(t,\mu,\lambda)=\int_0^\infty\Psi'_k(\mu,E,t)g(E)h(E)^{\lambda-1}dE, \quad (3.1)$$

or equivalent

$$\bar{\Psi}_k(t,\mu,\lambda)=\int_0^\infty\Psi_k(\mu,E,t)h(E)^{-\lambda}dE\equiv\mathfrak{M}\Psi_k(\mu,E,t), \quad (3.2)$$

where $\Psi'_k(\mu,E,t,)=\Psi_k(\mu,E,t)/g(E)$.

The transformation $\mathfrak{M}$ always exists provided,

$$\int_0^\infty|\Psi_k(\mu,E,t)|dE<\infty \quad (3.3)$$

Its inverse transformation is given as

$$\Psi'_k(t,\mu,E)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\bar{\Psi}_k(t,\mu,\lambda)h(E)^{-\lambda}d\lambda, \quad (3.4)$$

where integration path in the complex λ-plane must be to the right of all λ-singularities.

Later we need a "convolution" theorem for the $\mathfrak{M}$-transformation:

Theorem 3.1. If $\bar{F}(\lambda)$ and $\bar{G}(\lambda)$ are the $\mathfrak{M}$-transform of $F(E)$ and $G(E)$, then the inverse-transform of the product $\bar{F}(\lambda):\bar{G}(\lambda)$ is

$$\mathfrak{M}^{-1}[\bar{F}(\lambda)\bar{G}(\lambda)]=g(E)\int_v^1 F'\left(\frac{v}{w}\right)G'(w)\frac{dw}{w},$$

where $F'=F(E)/g(E)$ $G'(E)=G(E)/g(E)$

$v=h(E)$ $w=h(E')$

Proof: Defining $F'(E)=F(E)/g(E)$,

$G'(E)=G(E)/g(E)$,

we can rewrite the transformation $\mathfrak{M}$ as:

$$\bar{F}(\lambda)=\int_0^1 F'(v)v^{\lambda-1}dv,$$

$$\bar{G}(\lambda)=\int_0^1 G'(v)v^{\lambda-1}dv,$$

where we used the transformation defined in Eq. (3.1), and

$$v=h(E). \tag{3.5}$$

As we know from Eq. (3.2), this is nothing but a classical *Mellin*-transform. Then, a classical "convolution theorem" for the Mellin transform[9] states that:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\bar{F}(\lambda)\bar{G}(\lambda)v^{-\lambda}d\lambda=\int_0^1 F\left(\frac{v}{w}\right)G(w)\frac{dw}{w}$$

Notice $F'(v)=G'(v)=0$ for $v>1$.

So

$$\int_0^1 F'\left(\frac{v}{w}\right)G'(w)\frac{dw}{w}=\int_v^1 F'\left(\frac{v}{w}\right)G'(w)\frac{dw}{w}.$$

And finally, taking into account the factor $g(E)$:

$$\mathfrak{M}^{-1}[\bar{F}(\lambda)\bar{G}(\lambda)]=g(E)\int_v^1 F'\left(\frac{v}{w}\right)G'(w)\frac{dw}{w}.$$

Q.E.D.

4. Completeness of Eigenfunctions

With these preliminary investigation we are finally prepared to solve the Eq. (2.1) exactly to find its Green's function.

We write Eq. (2.1) once more with the explicitly given kernel.

$$\left(\frac{\partial}{\partial t}+ik\mu+1\right)\Psi_k(\mu E,t)=\frac{1}{2}\delta(t)S(E)$$
$$+\frac{c_F}{2}\int_{-1}^1 d\mu'\chi(E)\int_0^\infty \Psi_k(\mu',E',t)dE'$$
$$+\frac{c_i}{2}g(E)\int_E^\infty\frac{dE'}{h(E')}\int_{-1}^1\Psi_k(\mu,'E',t)d\mu'$$

Define an operator T by

$$T\phi(E)=c_F\chi(E)\int_0^\infty\phi(E)dE$$
$$+c_ig(E)\int_E^\infty\frac{\phi(E')}{h(E')}dE'. \tag{4.1}$$

We are considering function in the Banach space $L_1[0,\infty]$, such that $\phi(E)\in L_1[0,\infty]$, if and only if

$$||\phi||\equiv\int_0^\infty|\phi(E)|dE<\infty. \tag{4.2}$$

Such functions always possess the $\mathfrak{M}$-transform defined in Section 3:

$$\bar{\phi}(\lambda)=\int_0^\infty\phi(E)h(E)^{\lambda-1}dE. \tag{4.3}$$

So we can, in fact, consider a broader class: namely "tempered distribution.", which according to L. Schwartz[10], possess the $\mathfrak{M}$-transform. Now, let us look for eigenfunction of T.

$$T\phi(E)=\nu\phi(E) \tag{4.4}$$

We $\mathfrak{M}$-transform Eq. (4.1) and get

$$\nu\bar{\phi}(\lambda)=c_F\bar{\chi}(\lambda)\int_0^\infty\phi(E)dE+\frac{c_i}{\lambda}\bar{\phi}(\lambda), \tag{4.5}$$

where $\bar{\chi}(\lambda)=\int_0^\infty\chi(E)h(E)^{\lambda-1}dE,$

$$\bar{\chi}(1)=1. \tag{4.6}$$

Let us note that

$$\int_0^\infty\phi(E)dE=\bar{\phi}(1). \tag{4.7}$$

Using Eq. (4.7), Eq. (4.5) becomes

$$\nu\bar{\phi}(\lambda)=c_F\bar{\chi}(\lambda)\bar{\phi}(1)+\frac{c_i}{\lambda}\bar{\phi}(\lambda). \tag{4.8}$$

Solutions of (4.5) belong to two classes:

(I) Eigenfunctions $\phi(E)$ such that $\bar{\phi}(1)\neq 0$ or

$$\int_0^\infty\phi(E)dE\neq 0.$$

Then the solution of (4.3) is

$$\bar{\phi}(\lambda)=\frac{c_F\bar{\chi}(\lambda)\bar{\phi}(1)}{\nu-c_i/\lambda}. \tag{4.9}$$

But Eq. (4.9) must be verified for *all* values of λ; hence, for $\lambda=1$, it must yield an identity:

$$\bar{\phi}(1)=\frac{c_F}{\nu-c_i}\bar{\phi}(1). \tag{4.10}$$

Hence

$$\nu=c_F+c_i \tag{4.11}$$

So we have a unique eigenvalue $\nu=c_F+c_i$, to which corresponds a unique eigenfunction

$$F(\lambda)=\frac{c_F\bar{\chi}(\lambda)}{c_F+c_i(1-1/\lambda)}. \tag{4.12}$$

In fact, we have already implicitly solved the direct Eq. (4.4) in Section 2: using the solution (2.21) we obtain the direct expression for $H(E)$:

$$H(E)=\frac{c_F}{c_F+c_i}\chi(E)+\frac{c_i}{c_F+c_i}g(E)\times h(E)^{-\frac{c_i}{c_F+c_i}}\int_E^{\infty}\frac{c_F}{c_F+c_i}\frac{\chi(E')}{h(E')^{\frac{c_F}{c_F+c_i}}}dE' \tag{4.13}$$

(II) Eigenfunctions $\phi(E)$ such that $\bar{\phi}(1)=0$ or $\int_0^{\infty}\phi(E)dE=0.$

Then Eq. (4.8) reduces to

$$\nu\bar{\phi}(\lambda)=\frac{c_i}{\lambda}\bar{\phi}(\lambda). \tag{4.14}$$

The solution forms a *continuum* of "pseudo-eigenfunctions" such that

$$\bar{\phi}_\nu(\lambda)=\delta(\lambda-\lambda_0) \qquad \text{with } \lambda_0\neq 1,$$
$$\nu=c_i/\lambda_0$$
$$\phi_\nu(E)=\frac{g(E)}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\lambda\delta(\lambda-\lambda_0)h(E)^{-\lambda}$$
$$=g(E)h(E)^{-\lambda_0}. \tag{4.15}$$

Next we prove a completeness theorem for these two classes of functions.

Theorem 4.1. The discrete eigenfunction $H(E)$ defined by Eq. (4.13) and the "pseudo-eigenfunctions" defined by Eq. (4.15) form a *complete set* for functions $\in L_1[0,\infty]$.

Proof: Let $\phi(E)\in L_1\,[0,\infty]$. The completeness theorem can be stated as:

$$\phi(E)=JH(E)+\frac{g(E)}{2\pi i}\int_{c-i\infty}^{c+i\infty}A(\lambda_0)h(E)^{-\lambda_0}\lambda d\lambda_0$$

with $A(1)\equiv 0$; J and $A(\lambda_0)$ being the unknown expansion coefficient associated respectively with the discrete eigenfunction $H(E)$ and the continuum eigenfunction $h(E)^{-\lambda_0}$. Then we have obviously $J=\int_0^{\infty}\phi(E)dE$. Defining $\Gamma(E)=\phi(E)-JH(E)$, we have

$$\int_0^{\infty}\Gamma(E)dE\equiv 0.$$

(Since $\int_0^{\infty}H(E)dE\equiv 1$). Then, we must prove that we can write

$$\Gamma(E)=\frac{g(E)}{2\pi i}\int_{c-i\infty}^{c+i\infty}A(\lambda_0)h(E)^{-\lambda_0}d\lambda_0.$$

But, applying an inverse $\mathfrak{M}$-transformation, we get, since $\Gamma(E)$ is $\mathfrak{M}$-transformable;

$$A(\lambda_0)=\int_0^{\infty}\Gamma(E)h(E)^{\lambda_0-1}dE.$$

Our expansion coefficients are therefore known. We have only to verify that $A(1)\equiv 0$, which is true since;

$$A(1)=\int_0^{\infty}\Gamma(E)dE\equiv 0 \qquad \text{Q.E.D.}$$

Hence the eigenfunctions of the continuous energy transfer operator form a complete set, and that they can be classified into two groups; (1) one *discrete* regular eigenfunction, (2) a continuous set of (pseudo) eigenfunctions of *null measure*.

5. Solution of the complete equation

Let us now consider inhomogeneous term (source term) $S(E)$ of Eq. (4.1). In general, the source term is not proportional to $H(E)$. The idea is then *to decompose the actual source through an expansion using the complete set of energy-eigenfunctions* by theorem 4.1, since *any* source $S(E)\in L_1[0,\infty]$.

$$S(E)=JH(E)+\Gamma(E), \tag{5-1}$$

where $J=\int_0^{\infty}S(E)dE,$

clearly $\int_0^{\infty}\Gamma(E)dE=0.$

Then, from the completeness theorem 4.1

$$\Gamma(E)=\frac{g(E)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\bar{\Gamma}(\lambda)h(E)^{-\lambda}d\lambda \tag{5.2}$$

with $\bar{\Gamma}(\lambda)=\int_0^{\infty}\Gamma(E)h(E)^{\lambda-1}dE$

$\bar{\Gamma}(1)\equiv 0$

Then transport equation (2.1) is linear; so its solution can be expressed as the one speed solution due to the component $JH(E)$ plus the solution due to a source $\Gamma(E)$ of zero measure.

Call the former solution $H(E)\phi_E(t,\mu)$, and the latter $\phi_{tr}(t,\mu,E)$:

$$\Psi_k(t,\mu,E)=H(E)\phi_E(t,\mu)+\phi_{tr}(t,\mu,E) \tag{5.3}$$

$\phi_E(t,\mu)$ obeys

$$\left(\frac{\partial}{\partial t}+ik\mu+1\right)\phi_E(t,\mu)=\frac{c_F+c_i}{2}\int_{-1}^{1}\phi_E(t,\mu')d\mu'+\frac{J\delta(t)}{2}. \tag{5.4}$$

As to $\phi_{tr}(t,\mu,E)$, it obeys

$$\left(\frac{\partial}{\partial t}+ik\mu+1\right)\tilde{\phi}_k(t,\mu,\lambda)$$
$$=\frac{c_i}{2\lambda}\int_{-1}^{1}\tilde{\phi}_k(t,\mu'\lambda)d\mu'+\bar{\Gamma}(\lambda)\frac{\delta(t)}{2}. \quad (5.4)$$

Since both Eqs (5.3) and (5.4) are a well-known "monokinetic" equations. We can immediately write down the complete solution.

Defining the Green's function as

$$G_k(t,E)=\int_{-1}^{1}\Psi_k(t,E,\mu)d\mu, \quad (5.5)$$

we have

$$G_k(t,E)=\frac{e^{-t}}{2}\Big\{JH(E)\Big[\frac{exp(-i\alpha_0 kt)}{N\alpha_0(k)}+\int_{-1}^{1}\frac{exp(-i\alpha kt)}{N(\alpha,k)}d\alpha\Big]$$
$$+\frac{g(E)}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\lambda\bar{\Gamma}(\lambda)h(E)^{-\lambda}\Big[\frac{exp(-i\alpha_{0k}(\lambda)kt)}{N\alpha_0(k,\lambda)}+\int_{-1}^{1}\frac{exp(-i\alpha(\lambda)kt)}{N(\alpha,k,\lambda)}d\alpha\Big]\Big\},$$

where

$$\alpha_{0k}=i/tan(k/c_i+c_F),\ \alpha_{0k}(\lambda)=i/tan\ k\lambda/c_i$$
$$N\alpha_0(k)=-\frac{(c_i+c_F)^2}{2k^2}\ \frac{1}{1-\alpha_{0k}^2},$$
$$N\alpha_0(k,\lambda)=\frac{-c_i^2}{2k^2\lambda^2}\ \frac{1}{1-\alpha_{0k}(\lambda)^2},$$
$$N(\alpha,k)=\Big[1-\frac{i(c_F+c_i)}{k}tank^{-1}\alpha\Big]^2-\Big[\frac{(c_F+c_i)\pi}{2k}\Big]^2,$$
$$N(\alpha,k,\lambda)=\Big[1-\frac{ic_i}{k\lambda}tanh^{-1}\alpha\Big]^2-\Big[\frac{c_i\pi}{2k\lambda}\Big]^2.$$

For $\lambda<1$, we note immediately that $\alpha_o(\lambda)<\alpha_o$, we find the unique asymptotically dominant decay constant as $1-c$, where $-c=i\alpha_{ok}$ which is negative real number, if $k>0$.

6. Conclusion

We have found a complete solution of the integro-differential equation of neutron transport problem. Energy-time separable mode was found to be asymptotically dominant with a unique decay constant $(1-c)$ which will persist after a long period of time.

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