Expansions for Hyperbolic Orbit

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Abstract

A hyperbolic orbit is expanded in terms of F analogous to the eccentric anomaly of an elliptical orbit: $r^p \sin qv$ and $r^p \cos qv$ are expressed in terms of F. The true anomaly v is expressed in terms of F, and F in terms of v.

I. TRANSFORMATION OF VARIABLES

Before preceeding to the development in series relating to the hyperbolic orbit, we introduce several variables for transformation:

$$\phi=e^{iv}, \quad \chi=e^{-F} \qquad (\text{I}-1)$$

where $i^2=-1$ and e is the base of Napierian logarithms. v is true anomaly and F is analogous to the eccentric anomaly of an elliptical orbit.

We write the eccentricity of the hyperbolic orbit as

$$\varepsilon=\cosh\varphi \qquad (\text{I}-2)$$

and a quantity, β as

$$\beta=i\tanh\frac{\varphi}{2} \qquad (\text{I}-3)$$

From the above two relations,

$$\varepsilon=\frac{1-\beta^2}{1+\beta^2} \qquad (\text{I}-4)$$

and

$$\beta=i\sqrt{\frac{\varepsilon-1}{\varepsilon+1}} \qquad (\text{I}-5)$$

In the hyperbolic orbit there is a relation between v and F;

$$\tan\frac{v}{2}=\sqrt{\frac{\varepsilon+1}{\varepsilon-1}}.\tanh\frac{F}{2} \qquad (\text{I}-6)$$

By using the variables defined in (I—1) and (I—5), the relation(I—6) is expressed as

$$\frac{\phi-1}{\phi+1}=-\frac{1}{\beta}\cdot\frac{1-\chi}{1+\chi}$$

From this relation

$$\phi=\frac{\chi(1+\beta)-(1-\beta)}{\chi(\beta-1)+(1+\beta)} \qquad (\text{I}-7)$$

and

$$\chi=\frac{(\beta-1)-\phi(\beta+1)}{\phi(\beta-1)-(\beta+1)} \qquad (\text{I}-8)$$

The equation of the hyperbolic orbit given by

$$r=a(\varepsilon\cosh F-1)$$

then can be expressed in terms of new variables:

$$r=\frac{a[(\chi-1)+\beta(\chi+1)]\cdot[(\chi-1)-\beta(\chi+1)]}{2\chi(1+\beta^2)} \qquad (\text{I}-9)$$

II. EXPANSION OF v IN TERMS OF F

Eq(I—7) is written as

$$\phi=\frac{\beta-1}{\beta+1}\cdot\frac{1-\alpha\chi}{1-\alpha^{-1}\chi} \qquad (\text{II}-1)$$

where

$$\alpha\equiv\frac{1+\beta}{1-\beta} \qquad (\text{II}-2)$$

By taking logarithms and expanding in series, Eq(II—1) becomes

$$iv=l_n\left(\frac{\beta-1}{\beta+1}\right)-\sum_{n=1}^{\infty}\frac{1}{n}e^{-nF}(e^{in\kappa}-e^{-in\kappa}) \qquad (\text{II}-3)$$

where from (II—2),

$$\alpha^n-\alpha^{-n}=e^{in\kappa}-e^{-in\kappa}$$

and

$$\kappa \equiv \cos^{-1}\frac{1}{\varepsilon}$$

From(I —5)

$$\frac{\beta-1}{\beta+1} = e^{i(\pi-2\theta)}$$

where

$$\theta = \tan^{-1}\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}$$

and

$$2\tan^{-1}\sqrt{\frac{\varepsilon-1}{\varepsilon+1}} = \cos^{-1}\frac{1}{\varepsilon}$$

Finally the true anomaly v in (II —3) then is expanded in the following series:

$$v = \pi - \cos^{-1}\frac{1}{\varepsilon} - \sin\left(\cos^{-1}\frac{1}{\varepsilon}\right) \cdot \sum_{n=1}^{\infty}\frac{2}{n} \cdot U_{n-1}\left(\frac{1}{\varepsilon}\right) e^{-nF} \qquad \text{(II—4)}$$

where U_n is a Chebyshev polynomial of the second kind defined by

$$U_n(x) = \frac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)}$$

or by Jacobic polynomial, $P_n^{(\alpha,\beta)}(x)$ (Abramowitz and Stegun, 1964),

$$U_n(x) = \frac{(n+1)!\sqrt{\pi}}{2\Gamma\left(n+\frac{3}{2}\right)} P_n^{(\frac{1}{2},\frac{1}{2})}(x)$$

As r goes to infinity, F also goes to infinity while v approaches a finite value, $\pi - \cos^{1}\frac{1}{\varepsilon}$, which is an angle of asymptotic line of the hyperbolic orbit.

III. EXPANSION OF *F* IN TERMS OF *v*

From(I —8)

$$\chi = \frac{1-\beta}{1+\beta} \cdot \frac{1+\alpha\phi}{1+\alpha^{-1}\phi} \qquad \text{(III—1)}$$

where α is defined in (II —2).

Taking logarithms in Eq(III —1) and expanding in series, we have

$$-F = -i\cos^{-1}\frac{1}{\varepsilon} + 2i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\left(n\cos^{-1}\frac{1}{\varepsilon}\right) \cdot \cos nv - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\left(n\cos^{-1}\frac{1}{\varepsilon}\right) \cdot \sin nv \qquad \text{(III—2)}$$

The real part in (III —2) presents the required series expansion of F in terms of v:

$$F = 2\sin\left(\cos^{-1}\frac{1}{\varepsilon}\right) \cdot \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \cdot U_{n-1}\left(\frac{1}{\varepsilon}\right) \cdot \sin nv \qquad \text{(III—3)}$$

The imaginary part yields a relation of series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \cdot \sin nk \cdot \cos nv = \frac{k}{2}$$

where $|k| < 1$ and $|v| < \pi$.

IV. EXPANSIONS OF $r^p\cos qv$ AND $r^p\sin qv$ IN TERMS OF *F*

From (I —7) and (I —9)

$$r^p\phi^q = \left(\frac{a}{2\chi}\right)^p \cdot \frac{[(1-\chi)-\beta(1+\chi)]^p \cdot [(1-\chi)+\beta(1+\chi)]^p \cdot [\beta(1+\chi)-(1-\chi)]^q}{(1+\beta^2)^p \cdot [(1-\chi)+\beta(1+\chi)]^q}$$

$$= \left(\frac{a}{2}\right)^p(-1)^q \cdot \frac{(1+\beta)^{2p}}{\chi^p(1+\beta^2)^p}\left[\frac{1-\beta}{1+\beta}-\chi\right]^{p+q} \cdot \left[1-\chi\left(\frac{1-\beta}{1+\beta}\right)\right]^{p-q}\left(\frac{a\varepsilon}{2}\right)^p(-1)^q\chi^{-p}$$

$$\cdot \sum_{l=0}\sum_{m=0}\binom{p+q}{l}\binom{p-q}{m}(-1)^{l+m} \cdot 2^{q-l+m} \cdot \varepsilon^{q-l+m} \cdot (\varepsilon+1)^{l-m-q} \cdot (1+\beta)^{-2q+2l-2m} \cdot \chi^{l+m} \qquad \text{(IV—1)}$$

In the above series, if p and q are positive integers and $p \geq q$, (IV—1) has finite terms. Otherwise, it has infinite terms.

Since $\varepsilon = \frac{1-\beta^2}{1+\beta^2}$ and $(1-\beta) = \frac{2\varepsilon}{1+\varepsilon} \cdot \frac{1}{1+\beta}$, the series(IV—1) becomes

$$r^p\phi^q = a^p(-1)^q 2^{q-p}\varepsilon^p\chi^{-p}A^{-q}\sum_{l=0}\sum_{m=0}\binom{p+q}{l}\binom{p-q}{m} \cdot (-1)^{l+m} \cdot \chi^{l+m} \cdot 2^{m-l} \cdot A^{l-m} \cdot \sum_{n=0}^{\infty}\binom{-2q+2l-2m}{n}$$

$$\cdot (iB)^n = \left(\frac{a\varepsilon}{2\chi}\right)^p \sum_{k=0}\sum_{m=0}^{k}\binom{p+q}{k-m}\binom{p-q}{m} \cdot (-1)^{k+q}\chi^k\left(\frac{A}{2}\right)^{k-2m-q}$$

$$\cdot\sum_{n=0}^{\infty}\binom{-2q+2k-4m}{n}(iB)^n \qquad (\mathrm{IV}—2)$$

where $A\equiv 1+\frac{1}{\varepsilon}$ and $B\equiv\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}$.

The first combination in Eq(IV—2)can be expressed as

$$\binom{p+q}{k-m}=\binom{p+q}{k}\cdot\frac{k}{p+q-k+1}\cdot\frac{k-1}{p+q-k+2}\cdots\cdots\frac{k-m+1}{p+q-k+m}$$

Thus the multiplication of two combinations in Eq(IV—2) becomes

$$\sum_{m=0}^{k}\binom{p+q}{k-m}\binom{p-q}{m}=\binom{p+q}{k}\cdot\Big[1+\frac{k}{p+q-k+1}\cdot\frac{p-q}{1}+\frac{k(k-1)}{(p+q-k+1)(p+q-k+2)}\cdot\frac{(p-q)(p-q-1)}{1\cdot 2}+\cdots\cdots+\frac{k(k-1)\cdots\cdots(k-m+1)}{(p+p-k+1)(p+q-k+2)\cdots(p+q-k+m)}\cdot\frac{(p-q)(p-q-1)\cdots(p-q-m+1)}{m!}+\cdots\cdots+\frac{k(k-1)\cdots 1}{(p+q-k+1)\cdots(p+q)}\cdot\frac{(p-q)(p-q-1)\cdots(p-q-k+1)}{k!}\Big]$$

$$=\binom{p+q}{k}\cdot\sum_{m=0}^{k}\frac{(-k)_m(q-p)_m}{(p+q-k+1)_m m!} \qquad (\mathrm{IV}—3)$$

where in general, $(\alpha)_m=\alpha(\alpha+1)\cdots(\alpha+m-1)$ and $(\alpha)_0=1$.

Using Eq(IV—3), we rewrite (IV—2) as

$$r^p\phi^q=\left(\frac{a\varepsilon}{2}\right)^p(-1)^q\left(\frac{2}{A}\right)^q\sum_{k=0}\binom{p+q}{k}\cdot(-1)^k\chi^{k-p}\left(\frac{A}{2}\right)^k\cdot\sum_{m=0}^{k}\left(\frac{2}{A}\right)^{2m}\cdot\frac{(-k)_m(q-p)_m}{(p+q-k+1)_m m!}\cdot\sum_{n=0}^{\infty}\binom{-2q+2k-4m}{n}(iB)^n \qquad (\mathrm{IV}—4)$$

The both sides of Eq (IV—4) include real and imaginary parts.

The left side of Eq(IV—4) is written, from(I—1), as

$$r^p\phi^q=r^p(\cos qv+i\sin qv)$$

Hence the real part of Eq(IV—4) is

$$r^p\cos qv=\left(\frac{a\varepsilon}{2}\right)^p(-1)^q\cdot\left(\frac{2}{A}\right)^q\cdot\sum_{k=0}\binom{p+q}{k}\cdot(-1)^k\left(\frac{A}{2}\right)^k\cdot\chi^{k-p}\cdot\sum_{m=0}^{k}\left(\frac{2}{A}\right)^{2m}\cdot\frac{(-k)_m(q-p)_m}{(p+q-k+1)_m m!}\cdot\sum_{n=0}^{\infty}\binom{-2q+2k-4m}{2n}(B^2)^n \qquad (\mathrm{IV}—5)$$

and the imaginary part is

$$r^p\sin qv=\left(\frac{a\varepsilon}{2}\right)^p(-1)^q\left(\frac{2}{A}\right)^q\cdot\sum_{k=0}\binom{p+q}{k}\cdot(-1)^k\left(\frac{A}{2}\right)^k\cdot\chi^{k-p}\cdot\sum_{m=0}^{k}\left(\frac{2}{A}\right)^{2m}\cdot\frac{(-k)_m(q-p)_m}{(p+q-k+1)_m m!}\cdot\sum_{n=0}^{\infty}\binom{-2q+2k-4m}{2n+1}(-B^2)^n\cdot B \qquad (\mathrm{IV}—6)$$

The last serieses in Eqs(IV—5) and (IV—6) are expressed as

$$\sum_{n=0}^{\infty}\binom{-2q+2k-4m}{2n}(-B^2)^n=\sum_{n=0}^{\infty}\frac{(q-k+2m)_n\cdot\left(q-k+2m+\frac{1}{2}\right)_n}{n!\left(\frac{1}{2}\right)_n}(-B^2)^n$$

$$={}_2F_1\left(q-k+2m,q-k+2m+\frac{1}{2},\ \frac{1}{2};\ -B^2\right)$$

$$=\left(\frac{2\varepsilon}{1+\varepsilon}\right)^{k-q-2m}\cdot T_{k-q-2m}\left(\frac{1}{\varepsilon}\right) \qquad (\mathrm{IV}—7)$$

and

$$\sum_{n=0}^{\infty}\binom{-2q+2k-4m}{2n+1}(-B^2)^n=\sum_{n=0}^{\infty}\frac{(-q+k-2m)(q-k+2m+1)_n\left(q-k+2m+\frac{1}{2}\right)_n}{\frac{1}{2}\cdot\left(\frac{3}{2}\right)_n\cdot n!}\cdot(-B^2)^n=2(-q+k-2m)\cdot{}_2F_1\left(q-k+2m+\frac{1}{2},\ q-k+2m+1,\frac{3}{2};-B^2\right)$$

$$=2\left(\frac{2\varepsilon}{1+\varepsilon}\right)^{-q+k-2m-1}\cdot\frac{(-q+k-2m)!}{\left(\frac{3}{2}\right)_{-q+k-2m-1}}\cdot P^{(\frac{1}{2},\frac{1}{2})}_{-q+k-2m-1}\left(\frac{1}{\varepsilon}\right)$$

$$=2\left(\frac{2\varepsilon}{1+\varepsilon}\right)^{-q+k-2m-1}\cdot\frac{\varepsilon^2}{\varepsilon^2-1}\left[T_{-q+k-2m-1}\left(\frac{1}{\varepsilon}\right)-\frac{1}{\varepsilon}T_{-q+k-2m}\left(\frac{1}{\varepsilon}\right)\right] \qquad (\mathrm{IV}—8)$$

In (IV—7) and (IV—8), ${}_2F_1$ denotes a hypergeometric function and T_n is a

Chebyshev polynomial of the first kind defined by

$$T_n(1-2x)={}_2F_1\left(-n,n,\frac{1}{2};x\right)=\cos(n\cos^{-1}x)$$

In deriving the above serieses, the following relations were used:

$${}_2F_1(a,b,c;x)=(1-x)^{-a}\cdot{}_2F_1\left(a,c-b,c;\frac{x}{x-1}\right)$$

$${}_2F_1(-n,\alpha+1+\beta+n,\alpha+1;x)=\frac{n!}{(\alpha+1)_n}\cdot P_n^{(\alpha,\beta)}(1-2x)$$

$$U_n(x)=\frac{1}{1-x^2}\left[xT_{n+1}(x)-T_{n+2}(x)\right]$$

$$U_n(x)=T_n(x)+xU_{n-1}(x)$$

In (Ⅳ—5) and (Ⅳ—6),

$$\binom{p+q}{k}\frac{(-k)_m(q-p)_m}{(p+q-k+1)_m m!}=\frac{(p+q-k+1)_k(-k)_m(q-p)_m}{k!(p+q-k+1)_m m!}=\frac{1}{(k-m)!m!}(p-q-m+1)_m\cdot(p+q-k+m+1)_{k-m} \qquad (\text{IV}-9)$$

where $k\geq m$.

By (Ⅳ—7), (Ⅳ—8) and (Ⅳ—9), *Eqs*(Ⅳ—5) and (Ⅳ—6) reduce to the simple forms expressed in terms of F:

$$r^p\cos qv=\left(\frac{a\varepsilon}{2}\right)^p(-1)^q\sum_{k=0}\sum_{m=0}^{k}(-1)^k\cdot\frac{(p+q-k+m+1)_{k-m}(p-q-m+1)_m}{(k-m)!\ m!}\cdot T_{-q+k-2m}\left(\frac{1}{\varepsilon}\right)\cdot e^{-(k-p)F} \qquad (\text{IV}-10)$$

$$r^p\sin qv=\left(\frac{a\varepsilon}{2}\right)^p(-1)^q\sqrt{\frac{\varepsilon^2}{\varepsilon^2-1}}\sum_{k=0}\sum_{m=0}^{k}(-1)^k\cdot\frac{(p+q-k+m+1)_{k-m}(p-q-m+1)_m}{(k-m)!\ m!}\cdot\left[T_{-q+k-2m-1}\left(\frac{1}{\varepsilon}\right)-\frac{1}{\varepsilon}T_{-q+k-2m}\left(\frac{1}{\varepsilon}\right)\right]\cdot e^{-(k-p)F} \qquad (\text{IV}-11)$$

Some particular cases are as follows:

$$r^n=\left(\frac{a\varepsilon}{2}\right)^n\cdot\sum_{k=0}^{2n}\sum_{m=0}^{k}(-1)^k\cdot\frac{(-k+m+n+1)_{k-m}(-m+n+1)_m}{(k-m)!m!}\cdot T_{k-m}\left(\frac{1}{\varepsilon}\right)\cdot e^{-(k-n)F}\ :(n=0,1,2,\cdots\cdots)$$

$$r^{-n}=\left(\frac{a\varepsilon}{2}\right)^{-n}\cdot\sum_{k=0}^{\infty}\sum_{m=0}^{k}(-1)^k\cdot\frac{(k-m+1)_{m-1}(m+1)_{n-1}}{[(n-1)!]^2}\cdot T_{-k-2m}\left(\frac{1}{\varepsilon}\right)\cdot e^{-(k+n)F}\ :(n=1,2,3,\cdots\cdots)$$

$$\cos nv=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{n(k-m+1)_{n-1}}{(m-n)_{n+1}}\cdot T_{k-2m+n}\left(\frac{1}{\varepsilon}\right)\cdot e^{-kF}$$

$$\sin nv=-\sqrt{\frac{\varepsilon^2}{\varepsilon^2-1}}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{n(k-m+1)_{n-1}}{(m-n)_{n+1}}\cdot\left[T_{k-2m+n-1}\left(\frac{1}{\varepsilon}\right)-\frac{1}{\varepsilon}T_{k-2m+n}\left(\frac{1}{\varepsilon}\right)\right]\cdot e^{-kF}$$

$$\frac{\cos v}{r}=-\frac{2}{a\varepsilon}\sum_{m=0}^{\infty}(m+1)T_{m+1}\left(\frac{1}{\varepsilon}\right)\cdot e^{-(m+1)F}$$

$$\frac{\sin v}{r}=-\frac{2}{a\varepsilon}\sqrt{\frac{\varepsilon^2}{\varepsilon^2-1}}\sum_{m=0}^{\infty}(m+1)\left[T_{m+2}\left(\frac{1}{\varepsilon}\right)-\frac{1}{\varepsilon}T_{m+1}\left(\frac{1}{\varepsilon}\right)\right]\cdot e^{-(m+1)F}$$

The above two serieses are obtained from(Ⅳ—10) and (Ⅳ—11) by putting $k=m$, referring to *Eq*(Ⅳ—2).

In the similar way

$$r\cos v=-\left(\frac{a\varepsilon}{2}\right)\cdot\sum_{k=0}^{2}(-1)^k\frac{(-k+3)_k}{k!}\cdot T_{k-1}\left(\frac{1}{\varepsilon}\right)\cdot e^{-(k-1)F}=a(\varepsilon-\cosh F)$$

$$r\sin v=-\left(\frac{a\varepsilon}{2}\right)\cdot\sqrt{\frac{\varepsilon^2}{\varepsilon^2-1}}\cdot\sum_{k=0}^{2}(-1)^k\frac{(-k+3)_k}{k!}\cdot\left[T_{k-2}\left(\frac{1}{\varepsilon}\right)-\frac{1}{\varepsilon}T_{k-1}\left(\frac{1}{\varepsilon}\right)\right]\cdot e^{-(k-1)F}=a\sqrt{\varepsilon^2-1}\cdot\sinh F$$

$$\frac{\cos 2v}{r^3}=\frac{1}{3}\left(\frac{1}{a\varepsilon}\right)^3\cdot\sum_{k=0}^{\infty}\sum_{m=0}^{k}(m+1)_4T_{k-2m-2}\left(\frac{1}{\varepsilon}\right)\cdot e^{-(k+3)F}$$

$$\frac{\sin 2v}{r^3}=\frac{1}{3}\left(\frac{1}{a\varepsilon}\right)^3\sqrt{\frac{\varepsilon^2}{\varepsilon^2-1}}\sum_{k=0}^{\infty}\sum_{m=0}^{k}(m+1)_4\cdot\left[T_{k-2m-3}\left(\frac{1}{\varepsilon}\right)-\frac{1}{\varepsilon}T_{k-2m-2}\left(\frac{1}{\varepsilon}\right)\right]\cdot e^{-(k+3)F}$$

Reference

Abramowitz, M. and Stegun, I.A.; *Handbook of Mathematical Function*, Published by National Bureau of Standards, 1964