A CLASS OF MULTI-VALUED FUNCTIONS

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Let $e: X\times Y\to Z$ be a continuous function. If X, Y, Z are compact and Hausdorff, then the multi-valued function $F: Y\to Z$ defined by $F(y)=e(X\times\{y\})$ for $y\in Y$ is continuous [3]. The purpose of this note is to give necessary and sufficient conditions for a multi-valued function to be represented in such a fashion.

NOTATION: The topological closure of A is A^* and $\square$ is the empty set. Let F be a relation between Y and Z. If $y\in Y$, then $F(y)=\{z\in Z\mid (y,z)\in F\}$. If R is a subset of Z, then $F^{(-1)}(R)=\{y\in Y\mid F(y)\cap R\neq\square\}$ and $F^{[-1]}(R)=\{y\in Y\mid F(y)\subset R\}$. We shall follow the terminology of [1].

A multi-valued function $F: Y\to Z$ is a relation F between Y and Z such that $F(y)$ is not empty for all $y\in Y$. F is continuous if and only if (i) $F(y)$ is closed for all $y\in Y$ and (ii) if U is an open subset of Z, then $F^{(-1)}(U)$ and $F^{[-1]}(U)$ are open in Y. A trace of F is a continuous single-valued function $f: Y\to Z$ such that $f(y)\in F(y)$ for all $y\in Y$. A subfunction G of F is a continuous multi-valued function $G: Y\to Z$ such that $G(y)\subset F(y)$ for all $y\in Y$. For more complete treatments of relation theory the reader is referred to [2] and [4].

THEOREM. *Let Y and Z be compact, Hausdorff spaces and let $F: Y\to Z$ be a continuous multi-valued function. Then there exists a compact, Hausdorff space X and a continuous function $e: X\times X\to Z$ such that $e(X\times\{y\})=F(y)$ for all $y\in Y$ if and only if there is a family T of traces of F satisfying*:

(1) *If $y\in Y$ and $z\in F(y)$, then there is an $f\in T$ such that $f(y)=z$.*

(2) *T is closed in C where C is the set of all continuous functions from Y to Z with the compact-open topology.*

(3) *For every closed finite cover $\mathscr{M}$ of Z and for every $y\in Y$, there is a family $\{G_{M_y}\mid y\in F^{(-1)}(M),\ M\in\mathscr{M}\}$ of subfunction of f such that $G_{M_y}(y)=M\cap F(y)$ and every $f\in T$ is a trace of some G_{M_y}.*

PROOF. First we suppose that such an e and X exist. Define $g: X\to C$ by $g(x)=f$ if and only if $e(x,y)=f(y)$. Using the compact-open topology on C, it is easily shown that g is continuous. Thus if we let $T=g(X)$, then (1) and (2) are clear.

Let $\mathscr{M}$ be a closed cover of Z and $y \in Y$. If $M \in \mathscr{M}$ and $y \in F^{(-1)}(M)$, then let $A_M = \{x | e(x, y) \in M\}$. Since A_M is closed, the function $G_{My}: Y \to Z$ defined by $G_{My}(y') = e(A_M \times \{y'\})$ is continuous and a subfunction of F with $G_{My}(y) = F(y) \cap M$. If $f \in T$, then there is $x \in X$ and $M \in \mathscr{M}$ such that $g(x) = f$ and $f(g) \in M$. It follows that $x \in A_M$ so that f is a trace of G_{My}.

Next, we shall assume that a family T of traces of F satisfy (1), (2), and (3). Consider T as a subspace of C. Then $e: T \times Y \to Z$, which is defined by $e(f, y) = f(y)$, is coninouus and from (1) it follows that $e(T \times \{y\}) = F(y)$ for all $y \in Y$. It remains only to show that T is compact. Since by (2) T is a closed subspace of C, if T is equicontinuous, then it follows from the Ascoli Theorem ([1]; Theorem 17, pp. 233-234) that T is compact. Thus, to complete the proof of the theorem, it suffices to show T is equicontinuous.

Let $\mathscr{V}$ be a finite open cover of Z. Let $\mathscr{M}$ be a finite closed cover of Z, which refines $\mathscr{V}$, and fix $y \in Y$. Let $\{G_{My} | y \in F^{(-1)}(M)\}$ be a family of subfunctions of F given by (3). Let $U = \cap \{G_{My}{}^{[-1]}(V) | V \in \mathscr{V}$, *there is* $M \in \mathscr{M}$ *with* $y \in F^{(-1)}(M)$ *and* $M \subset V\}$. Thus U is an open neighborhood of y. Let $f \in T$, then there is $M \in \mathscr{M}$ and $V \in \mathscr{V}$ such that f is a trace of G_{My} and $M \subset V$. Thus, $f(u) \subset V$ since $y' \in U$ implies $y' \in G_{My}{}^{[-1]}(V)$ and $f(y') \in G_{My}(y') \subset V$.

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