HILBERT TRANSFORM IN SPACES Λ AND M

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1. Introduction.

Let f be a measurable function defined on some measure space S with measure m and f^* the non-increasing equimeasurable rearrangement of f onto $(0,\infty)$. (For definitions of these concepts see e.g. [2]). The spaces $\Lambda(\alpha,p)$, $\alpha>0$, $M(\alpha,p)$, $0\le\alpha\le1$, $1\le p<\infty$ are defined to consist of those measurable functions f on S for which the respective norms given by

$$\|f\|_{\Lambda(\alpha,\ p)}\equiv\left\{\alpha\int_0^\infty x^{\alpha-1}(f^*(x))^p\,dx\right\}^{1/p}$$

and

$$\|f\|_{M(\alpha,p)}\equiv\sup_{E\subset S}[m(E)]^{-\alpha}\left\{\int_E|f(x)|^p dx\right\}^{1/p},\quad m(E)<\infty,$$

are finite.

For $0<\alpha\le1$, $\Lambda(\alpha,p)$ and $M(\alpha,p)$ are Banach spaces and for $\alpha=1$, $\|f\|_{\Lambda(\alpha,p)}=\|f\|_p$, where the last norm is that of the usual Lebesgue spaces L_p.

It is well known (see e.g. [4]) that if $f\in L_p(-\infty,\infty)$, $1\le p<\infty$, then $\tilde{f}$ the Hilbert transform of f:

$$\tilde{f}(x)=\lim_{\varepsilon\to0^+}\frac{1}{\pi}\int_{|t-x|>\varepsilon}\frac{f(t)}{t-x}dt$$

exists a.e. and for $1<p<\infty$, $\|\tilde{f}\|_p\le A_p\|f\|_p$.

In this note we extend the previous result to functions in $\Lambda(\alpha,p)$ and $M(\alpha,p)$. Also, extending the definition of the Hilbert transform to certain singular integrals in E^n, similar results are obtained.

Throughout, A denotes a constant independent of f not necessarily the same each time.

In the sequel f^{**} denotes the integral mean of f^*, that is

$$f^{**}(x)=\frac{1}{x}\int_0^x f^*(t)dt,\quad x>0. \tag{1}$$

The following result due to R. O'Neil and G. Weiss [2] will be used frequently.

THEOREM 1. *If*

$$\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{1}{t}\right)dt<\infty$$

then $\tilde{f}$ *exists a.e. and for each* $s>0$

$$s\,\tilde{f}^{**}(s)\leq\frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt=\frac{2s}{\pi}\int_0^\infty\frac{f^{**}(t)}{(s^2+t^2)^{1/2}}dt.$$

A result similar to Theorem 1, with $\tilde{f}$ is replaced by a singular integral operator in E^n is given in the same paper [2]. This result will be used for the operators considered in § 3.

2. Main Results

THEOREM 2. *If* $f\in\Lambda(\alpha,p)$, $1<p<\infty$, $0<\alpha<p$, *and* q *and* β *satisfy* $\beta p=\alpha q$, $\alpha\leq\beta$, *then* $\tilde{f}$ *exists* a.e. *and*

$$\|\tilde{f}\|_{\Lambda(\beta,q)}\leq A\|f\|_{\Lambda(\alpha,p)}. \tag{2}$$

PROOF. Let $p+p'=pp'$ then Holder's inequality and a special case of [3, Theorem 2] yields

$$\alpha\int_0^\infty\frac{f^{**}(t)}{(1+t^2)^{1/2}}dt\leq\|f^{**}\|_{\Lambda(\alpha,p)}\left\{d\int_0^\infty t^{(1-\alpha)/(p-1)}(1+t^2)^{-p'/2}dt\right\}^{1/p'}$$

$$\leq\frac{p\alpha}{p-\alpha}\|f\|_{\Lambda(\alpha,p)}\left\{\int_0^R t^{(1-\alpha)/(p-1)}dt+\int_R^\infty t^{(1-\alpha)/(p-1)-p'}dt\right\}^{1/p'}<\infty.$$

The interchange of integration above is justified by Fubini's theorem, so that by Theorem 1, $\tilde{f}$. exists a.e.. From (1) we have the obvious inequality $f^*(s)\leq f^{**}(s)$, $s>0$, and by Theorem 1

$$\|\tilde{f}\|_{\Lambda(\beta,q)}\equiv\left\{\beta\int_0^\infty s^{\beta-1}[\tilde{f}^*(s)]^q ds\right\}^{1/q}\leq\left\{\beta\int_0^\infty s^{\beta-1}[\tilde{f}(s)]^q ds\right\}^{1/q}$$

$$\leq\left\{\beta\int_0^\infty s^{\beta-1}ds\left[\frac{2}{\pi s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt\right]^q\right\}^{1/q}$$

$$=\left(\frac{2}{\pi}\right)\left\{\beta\int_0^\infty\left[\int_0^\infty f^*(t)\,t^{(\alpha-1)/p}\right]\left[s^{(\beta-1)/q-1}t^{(1-\alpha)/p}\sinh^{-1}\left(\frac{s}{t}\right)dt\right]^q ds\right\}^{1/q}$$

Applying [1, Theorem 1] with $r=q$ and $v=(\beta-1)/q+(1-\alpha)/p$ we obtain (2).

THEOREM 3. *If* $f\in \Lambda(\alpha,1)$, $0<\alpha<1$, *then* $\tilde{f}$ *exists* a.e. *and*

$$\|\tilde{f}\|_{\Lambda(\alpha,1)}\leq A\|f\|_{\Lambda(\alpha,1)}.$$

PROOF. Since

$$\int_0^\infty \frac{f^{**}(t)}{(1+t^2)^{1/2}}dt=\int_0^\infty \frac{dt}{t(1+t^2)^{1/2}}\int_0^t f^*(s)ds$$

$$=\int_0^\infty s^{\alpha-1}f^*(s)ds\int_s^\infty \frac{s^{1-\alpha}dt}{t(1+t^2)^{1/2}}$$

$$\leq\|f\|_{\Lambda(\alpha,1)}\int_0^\infty \frac{dt}{t^\alpha(1+t^2)^{1/2}}<\infty,$$

the existence of $\tilde{f}$ a.e. follows from Theorem 1. Also

$$\|\tilde{f}\|_{\Lambda(\alpha,1)}\leq\left\{\alpha\int_0^\infty s^{\alpha-1}\tilde{f}^{**}(s)ds\right\}=\frac{2\alpha}{\pi}\int_0^\infty s^{\alpha-2}ds\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt$$

$$=\alpha\int_0^\infty f^*(t)dt\left\{\frac{2}{\pi}\int_0^\infty s^{\alpha-2}\sinh^{-1}\left(\frac{s}{t}\right)ds\right\}$$

$$=\|f\|_{\Lambda(\alpha,1)}\left\{\frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}u}{u^{2-\alpha}}du\right\}$$

$$=A\|f\|_{\Lambda(\alpha,1)},$$

which proves the result.

For the spaces $M(\alpha,p)$ we have the following:

THEOREM 4. *If* $f\in M(\alpha,p), 0<\alpha<\frac{1}{p}$, $1\leq p<\infty$, *then* $\tilde{f}$ *exists* a.e. *and*

$$\|\tilde{f}\|_{M(\alpha,p)}\leq A\|f\|_{M(\alpha,p)}.$$

PROOF. The proof of [3. Theorem 3] shows that

$$\int_0^\infty \frac{f^{**}(t)dt}{(1+t^2)^{1/2}}\leq\|f\|^*_{M(\alpha,p)}\int_0^\infty \frac{t^{\alpha-1/p}dt}{(1+t^2)^{1/2}}<\infty.$$

Therefore, by Theorem 1, $\tilde{f}$ exists, a.e.

Now let $E\subset S$ such that $m(E)=\eta<\infty$. Since $f^*(s)\leq f^{**}(s)$,

$$\eta^{-\alpha}\left\{\int_E|f(x)|^p dx\right\}^{1/p}\leq\eta^{-\alpha}\left\{\int_0^\eta[f^*(x)]^p dx\right\}^{1/p}\leq\eta^{-\alpha}\left\{\int_0^\eta[f^{**}(x)]^p dx\right\}^{1/p},$$

and by Theorem 1 and [3, Theorem 3]

$$\eta^{-\alpha}\left\{\int_E|\tilde{f}(x)|^p dx\right\}^{1/p}\leq\eta^{-\alpha}\left\{\int_0^\eta[\tilde{f}^{**}(x)]^p dx\right\}^{1/p}\leq\eta^{-\alpha}\left\{\int_0^\eta dx\left[\frac{2}{\pi}\int_0^\infty \frac{f^{**}(t)}{(x^2+t^2)^{1/2}}dt\right]^p\right\}^{1/p}$$

$$\leq \frac{2}{\pi}\|f\|_{M(\alpha,p)}\eta^{-\alpha}\Big\{\int_0^\eta dx\Big[\int_0^\infty \frac{t^{\alpha-1/p}}{(x^2+t^2)^{1/2}}\,dt\Big]^p\Big\}^{1/p}$$

$$=\frac{2}{\pi}\|f\|_{M(\alpha,p)}\eta^{-\alpha}\Big\{\int_0^\eta x^{\alpha p-1}\,dx\Big[\int_0^\infty \frac{u^{\alpha-1/p}du}{(1+u^2)^{1/2}}\Big]^p\Big\}^{1/p}$$

$$=A\|f\|_{M(\alpha,p)},$$

which proves the theorem.

COROLLARY. *If* $f\in M(\alpha, p)$, $0<\alpha$, $p\alpha\neq 1, 1\leq p<\infty$, *then* $\tilde{f}(x)$ *exists* a.e., *and*

$$\|\tilde{f}\|_{M(\alpha,p)}\leq A\|f\|_{M(\alpha,p)}.$$

PROOF. For $\alpha>\frac{1}{p}$, $M(\alpha,p)$ is void ([3, Theorem]) and the corollary reduces to Theorem 3.

3. Generalization.

Let $X, Y, \ldots$ denote the points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $\ldots$ of Euclidean n-space E^n, and $X', Y', \ldots$ be the points of the surface of the unit sphere Σ of E^n. That is $Y'=Y/|Y|$ where $|Y|=(y_1^2+\ldots+y_n^2)^{1/2}$. The volume, respectively, surface element in E^n and Σ is denoted by dY and dY'. We define $\hat{f}$, the singular integral operator with odd kernel by

$$\hat{f}(X)=\lim_{\varepsilon\to 0^+}\int_{|Y|>\varepsilon}\frac{\Omega(Y)}{|Y|^n}f(X-Y)dY$$

where Ω satisfies

(i) $\Omega(Y)=\Omega(Y')$

(ii) $\Omega(Y')=-\Omega(-Y')$ for all $Y'\in\Sigma$

(iii) $\|\Omega\|\equiv\int_\Sigma|\Omega(Y')|dY'<\infty$.

It is known [5] that $\hat{f}(X)$ exists when $f\in L_p(E^n)$, $1<p<\infty$ and $\|\hat{f}\|_p\leq A_p\|f\|_p$.

Considering now the spaces $\Lambda(\alpha,p)$ and $M(\alpha,p)$ corresponding to functions in E^n we obtain with the aid of the theorem [2] analogous to Theorem 1 the following.

THEOREM 5. *If* $f\in\Lambda(\alpha,p)$, $1<p<\infty$, $1<\alpha<p$ *and* α *and* β *satisfy* $\beta p=\alpha q$, $\alpha\leq\beta$ *then* $\hat{f}$ *exists* a.e. *and*

$$\|\hat{f}\|_{\Lambda(\beta, q)}\leq A\|f\|_{\Lambda(\alpha, p)}.$$

PROOF. By [2, Theorem 2]

$$\|\hat{f}\|_{\Lambda(\alpha,p)}=\left\{\alpha\int_0^\infty s^{\alpha-1}\left[[\hat{f}^*(s)\right]^p ds\right\}^{1/p}$$

$$\leq\left\{\alpha\int_0^\infty s^{\alpha-1}\left[\hat{f}^{**}(s)\right]^p ds\right\}^{1/p}$$

$$\leq\|\Omega\|\left\{\alpha\int_0^\infty s^{\alpha-1}ds\left[\frac{1}{s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt\right]^p\right\}^{1/p}.$$

Using the result of Okikiolu [1, Theorem 1], the result is obtained as in the proof of Theorem 2. The result corresponding to the Theorem 4 is :

THEOREM 6. *If* $f\in M(\alpha,p)$, $0<\alpha<\frac{1}{p}<1$ *then* $\hat{f}$ *exists* a.e. *and*

$$\|\hat{f}\|_{M(\alpha,p)}\leq A\|f\|_{M(\alpha,p)}.$$

The proof follows along the lines of that of Theorem 4.

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