THE EIGENVECTORS OF THE GRAM MATRIX AND THE COEFFICIENTS OF AN ORTHONORMAL SET OF FUNCTIONS

By K. W. Kaiser

Two sets of linearily independent functions are considered below. The first set is orthonormal and is obtained from the second by an orthonormalization process. The purpose of this paper is to show that a simple relationship exists between the coefficients of a certain orthonormal set and the eigenvectors of the Gram matrix.

More specifically, let ϕ_i denote a set of basis elements and $<x,y>$ an inner product for a function space Φ. An orthonormal set of functions, u_i, is constructed from the set ϕ_i by a linear transformation, A. Thus, u_i has the form

$$u_i=\sum_j a_{ij}\phi_j$$

and the following property.

$$<u_i, u_j> =\delta_{ij}$$

Many transformations, A, exist with this property.

It is shown that one particular set of coefficients, a_{ij}, is related to the set of eigenvectors of the Gram matrix in the following way:

$$a_{ij}=\frac{e_{ij}}{\sqrt{\lambda_i}}$$

e_{ij} is the j^{th} component of the i^{th} eigenvector of the Gram matrix of ϕ_i and λ_i is the corresponding eigenvalue. All other transformations are related to this one by an orthogonal transformation.

This relationship is developed in the following way: The elements ϕ_i are ordered in a vector denoted by $\underline{\phi}$.

$$\underline{\phi}^T=[\phi_1\phi_2\phi_3\cdots\cdots\cdots]$$

The elements u_i are ordered in a vector denoted by $\underline{u}$.

$$\underline{u}^T=[u_1u_2u_3\cdots]$$

The superscript, T, denotes the transpose of a vector or matrix. The coefficients a_{ij} are ordered in a matrix, A.

$$A=(a_{ij})$$

Equation (1) is rewritten in the above notation.

$$\underline{u}=A\underline{\phi}$$

The matrix, A, is partitioned into row vectors $\underline{a}_i^T$ that are the weighting coefficients of the i^{th} orthonormal function u_i; i.e.,

$$u_i=\underline{a}^T_i\underline{\phi}$$

The Gram matrix for the set ϕ_i is denoted by G. The elements of G are:

$$g_{ij}=\langle\phi_i,\phi_j\rangle$$

The requirement that the set u_i is orthonormal is written in the matrix notation.

$$\langle\underline{u},\underline{u}^T\rangle=I$$

I is the identity matrix.

Use of Equation (2) results in

$$\langle(A\underline{\phi}),(A\underline{\phi})^T\rangle=I$$

$$\langle A\underline{\phi},\underline{\phi}^T A^T\rangle=I$$

Since A is constant, it may be factored out of the inner product

$$A\langle\underline{\phi},\underline{\phi}^T\rangle A^T=I$$

Since

$$\langle\underline{\phi},\underline{\phi}^T\rangle=\mathrm{G}$$

Then

$$AGA^T=I$$

Let $\underline{v}$ be another set of orthonormal functions related to $\underline{u}$ by a transformation B.

$$\underline{v}=B\underline{u}$$

then

$$\underline{v}=BA\underline{\phi}$$

and

$$\langle BA\underline{\phi},\ (BA\underline{\phi})^T\rangle=I$$

or

$$BAGA^TB^T=I$$

From Equation (3), one obtains

$$BB^T=I$$

Thus, any other set of orthogonal functions v_i is derivable from the set u_i by an orthogonal transformation.

Since ϕ_i is a linearily independent set, then G is non-singular, real, symmetric, and positive definite; and there exists a set of eigenvalues λ_i and eigenvectors $\underline{e}_i$ with the properties

$$G\ \underline{e}_i=\lambda_i\underline{e}_i,\quad \underline{e}_i^T\underline{e}_j=\delta_{ij},\quad \lambda_i \text{ are posititive real.}$$

A matrix, S, is formed with the eigenvectors $\underline{e}_i$ of G as the columns. A diagonal matrix Λ is formed from the eigenvalues of G with elements $\lambda_i\delta_{ij}$. It is well known that G is related to S and Λ as follows:

$$S^TGS=\Lambda$$

Since λ_i are positive, then the matrix $\Lambda^{-1/2}$ with the elements $\frac{\delta_{ij}}{\sqrt{\lambda_i}}$ can be formed. Pre and post multiplication of equation (4) by $\Lambda^{-1/2}$ results in

$$\Lambda^{-1/2}\ S^TGS\Lambda^{-1/2}=I$$

or

$$(S\Lambda^{-1/2})^TG(S\Lambda^{-1/2})=I$$

Comparison of this last relation with equation (3) yields

$$A=\Lambda^{-1/2}S^T$$

In terms of $\underline{a}_i$ and $\underline{e}_i$, this relationship is

$$\underline{a}_i=\frac{\underline{e}_i}{\sqrt{\lambda_i}}$$

Consequently, the expression for the orthonormal functions is

$$u_i=\frac{\underline{e}_i^T}{\sqrt{\lambda_i}}\underline{\phi}$$

Since an orthogonal transformation of $S\Lambda^{-1/2}$ in equation (5) leaves the form of the equation unchanged, then A in equation (6) is unique to within an orthogonal transformation, as was already shown. Thus, all transformations A are of the below form:

$$A=B\Lambda^{-1/2}S^T$$

where

$$BB^T=I$$

The generalization of this result to other spaces is clearly possible.

Massachusetts Institute of Technology
Instrumentation Laboratory
Cambridge, Massachusetts, U. S. A.