A CHARACTERIZATION OF SYMMETRIC GENERALIZED PROXIMITY SPACES

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In this paper there is obtained a p-neighborhood characterization of symmetric generalized proximity spaces (introduced by Lodato in [1]) similar to the p-neighborhood characterization of ordinary proximity spaces(c. f. [2] page 193).

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DEFINITION 1. (Lodato) A *symmetric generalized proximity space* $\mathscr{P}$ on X is a subset of $P(X)\times P(X)$ such that:

L_1: $(A,B)\notin\mathscr{P}$ implies $(B,A)\notin\mathscr{P}$;

L_2: $(A,B)\notin\mathscr{P}$ implies $A\cap B=\phi$;

L_3: $(A,B)\notin\mathscr{P}$ and $(A,C)\notin\mathscr{P}$ implies $(A,(B\cup C))\notin\mathscr{P}$;

L_4: $(A,B)\in\mathscr{P}$ implies $A\neq\phi$ and $B\neq\phi$;

L_5: $(A,B)\in\mathscr{P}$ and $(\{b\},C)\in\mathscr{P}$ for all b in B implies that $(A,C)\in\mathscr{P}$.

THEOREM 1. *For a symmetric generalized proximity space on X the following are true*:

A: $(A,B)\in\mathscr{P}$, $A\subseteq C$ *and* $B\subseteq D$ *implies* $(C,D)\in\mathscr{P}$;

B: *For all x in X* $(\{x\},\{x\})\in\mathscr{P}$;

C: *For all A in $P(X)$* $(\phi,A)\notin\mathscr{P}$;

D: $(A,B)\in\mathscr{P}$ *or* $(A,C)\in\mathscr{P}$ *implies* $(A,B\cup C)\in\mathscr{P}$;

E: $A\cap B\neq\phi$ *implies* $(A,B)\in\mathscr{P}$;

The proof of this theorem is straightfoward.

DEFINITION 2. A set B in $P(X)$ is a p-neighborhood of a set A in $P(X)$ (notation: $A\Subset B$) iff $(A,(X-B))\notin\mathscr{P}$.

THEOREM 2. *In a symmetric generalized proximity sqace* $(X, \mathscr{P})$ *the following are equivalent. For all* A, B, C *in* $P(X)$:

A: $A \mathscr{P} B$ *and* $\{x\} \mathscr{P} C$ *for all* x *in* B *implies* $A \mathscr{P} C$;

B: $A \mathscr{P} B$ *and* $A \bar{\mathscr{P}} C$ *implies there exists* x *in* B *such that* $\{x\} \bar{\mathscr{P}} C$;

C: $A \bar{\mathscr{P}} C$ *implies* $A \bar{\mathscr{P}} B$ *or there exists* x *in* B *such that* $\{x\} \bar{\mathscr{P}} C$;

D: $A \Subset B$ *and* $A \mathscr{P} C$ *implies there exists* x *in* C *such that* $\{x\} \Subset B$;

E: $A \Subset B$ *implies* $A \Subset C$ *or there exists* x *in* $(X\text{-}C)$ *such that* $\{x\} \Subset B$.

The proof of this theorem is straightfoward.

We obtain a p-neighborhood characterization of symmetric generalized proximity spaces on X by the follo wing.

THEOREM 3. *The relation* $\Subset$ *in a symmetric generalized proximity space* $(X, \mathscr{P})$ *satisfies the following conditions*:

Q_1: $X \Subset X$;

Q_2: $A \Subset B$ *implies* $A \subseteq B$;

Q_3: $A \subseteq B \Subset C \subseteq D$ *implies* $A \Subset D$;

Q_4: $A \Subset B$ *and* $A \Subset C$ *implies* $A \Subset (B \cap C)$;

Q_5: $A \Subset B$ *implies* $(X\text{-}B) \Subset (X\text{-}A)$;

Q_6: $A \Subset B$ *implies* $A \Subset C$ *or there exists an* x *in* $(X\text{-}C)$ *such that* $\{x\} \Subset B$.

If $\mathscr{P}$ *is separated, then* Q_7: $\{x\} \Subset (X\text{-}\{y\})$ *iff* $x \neq y$. *Conversely, let a relation* $\Subset$ *satisfying* Q_1 *through* Q_6 *be defined on* $P(X)$. *Then* $\mathscr{P}$, *defined by* $A \bar{\mathscr{P}} B$ *iff* $A \Subset (X\text{-}B)$, *is a symmetric generalized proximity space on* X. *Furthermore,* B *is a* p*-neighborhood of* A *with respect to* $\mathscr{P}$ *iff* $A \Subset B$; *and if* Q_7 *is satisfied, then* $\mathscr{P}$ *is a separated proximity space on* X.

PROOF. Q_1: (by theorem 1(C)) $\phi \bar{\mathscr{P}} X$; so that (by L_1) $X \bar{\mathscr{P}} \phi$; so that $X \bar{\mathscr{P}} (-X)$; so that $X \Subset X$.

Q_2: Suppose $A \not\subseteq B$. Then $A \cap (X\text{-}B) \neq \phi$; so that (by theorem 1 (E)) $A \mathscr{P} (X\text{-}B)$. But $A \Subset B$ implies $A \bar{\mathscr{P}} (X\text{-}B)$.

Q_3: Suppose $A \not\Subset D$. Then $A \mathscr{P} (X\text{-}D)$. But $B \supseteq A$ and since $C \subseteq D$ we have that $(X\text{-}C) \supseteq (X\text{-}D)$; so that (by theorem 1 (A) and (E)) $B \mathscr{P} (X\text{-}C)$. But $B \Subset C$ implies $B \bar{\mathscr{P}} (X\text{-}C)$.

Q_4: $A \Subset B$ implies $A \bar{\mathscr{P}} (-B)$ and $A \Subset C$ implies $A \bar{\mathscr{P}} (-C)$; so that (by L_3) we have $A \bar{\mathscr{P}} ((-B) \cup (-C))$; so that $A \bar{\mathscr{P}} -(B \cap C)$; so that $A \Subset (B \cap C)$.

Q_5: Suppose not. Then $(X\text{-}B)\,\mathscr{P}\,A$ implies (by L_1) $A\,\mathscr{P}\,(X\text{-}B)$. But $A \Subset B$ implies $A\,\overline{\mathscr{P}}\,(X\text{-}B)$.

Q_6: Immediate (by theorem 2 (E)).

Q_7: $\{x\} \Subset (X\text{-}\{y\})$ implies $\{x\}\overline{\mathscr{P}}\{y\}$ implies $x \neq y$. $x \neq y$ implies $\{x\}\,\overline{\mathscr{P}}\,\{y\}$ implies $\{x\}\,\overline{\mathscr{P}}\,(X\text{-}(X-\{y\}))$ implies $\{x\} \subseteq (X\text{-}\{y\})$.

Proof of Converse. L_1: Suppose $A\,\overline{\mathscr{P}}\,B$. Then $A \Subset (\text{-}B)$; so that (by Q_4) $B = \text{-}(\text{-}B) \Subset (\text{-}A)$; so that $B\,\overline{\mathscr{P}}\,A$.

L_2: $A\,\overline{\mathscr{P}}B$ implies $A \Subset (\text{-}B)$ implies (by Q_2) that $A \subseteq (\text{-}B)$; so that $A \cap B = \phi$.

L_3: Suppose $A\overline{\mathscr{P}}\,B$ and $A\overline{\mathscr{P}}\,C$. Then $A \Subset (\text{-}B)$ and $A \Subset (\text{-}C)$ implies (by Q_4) that $A \Subset (\text{-}B) \cap (\text{-}C)$ implies $A \Subset \text{-}(B \cup C)$ implies $A\,\overline{\mathscr{P}}\,(B \cup C)$.

L_4: Let A be in $P(X)$. Then $A \subseteq X \Subset X$ (by Q_1). Hence $A \Subset X$ (by Q_3); so that $A\,\overline{\mathscr{P}}\,\phi$. Now suppose $A\,\mathscr{P}\,B$. Then $B \neq \phi$. But $B\,\mathscr{P}\,A$ (by L_1). Hence $A \neq \phi$.

L_5: Let $B' = (\text{-}B)$ and $C' = (\text{-}C)$. Then $A\,\overline{\mathscr{P}}\,(\text{-}B)$ implies by definition that $A \Subset B$; so that (by Q_6) $A \Subset C$ or there exists x in $(X\text{-}C)$ such that $\{x\} \Subset B$. So that $A\,\overline{\mathscr{P}}\,(\text{-}C)$ or there exists x in $(X\text{-}C)$ such that $\{x\}\,\overline{\mathscr{P}}\,(\text{-}B)$. Hence $A\overline{\mathscr{P}}\,B'$ implies $A\overline{\mathscr{P}}\,C'$ or there exists x in C' such that $\{x\}\,\overline{\mathscr{P}}\,B'$ and (by theorem 2 (C)) this is equivalent to L_5.

It is easily shown that the rest of the theorem is true.

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REFERENCES

[1] M. W. Lodato, *On topologically induced generalized proximity relations* π, Proc. Amer. Math. Soc., 15, No. 3 (1964), 417—422.

[2] W. J. Thron, *Topological structures*, Holt, Rinehart, and Winston, New York, 1966.