THE RANK OF THE PRODUCT OF TWO MATRICES

By Jin Bai Kim

1. It is well known that the rank of the product of two matrices cannot exceed the rank of either factor. The natural question can be raised: What is the rank of th product of two matrices? We shall give an answer to this question in Theorem 1 and we shall have an application of this Theorem 1.

2. Let F be a field, and $V(F)=V$ a vector space over F. Let $L(V)$ be the multiplicative semigroup of all linear transformations of V. With each element A of $L(V)$ we associate two subspaces of V:

(1) the range space $R(A)$ of A, consisting of all xA with x in V,

(2) the null space $N(A)$ of A, consisting of all y in V such that $yA=0$. $\rho(A)$ denotes the rank of the linear transformation A.

THEOREM 1. *If A and B are two linear transformations of a finite dimensional vector space $V(F)$. then $\rho(AB)=\rho(A)-dim(R(A)\cap N(B))$.*

PROOF. Let $\dim V(F)=n$. Let $\{z_i:i=1,2,\dots,m\}$ be a basis for the null space $N(A)$. We can supplement this basis by n-m vectors $\{x_i:\ i=1,2,\dots,n\text{-}m\}$ to obtain the basis $x_1,x_2,\dots,x_{n-m},z_1,z_2,\dots,z_m$ for $V(F)$. The vectors $x_1A,\ x_2A,\dots,x_{n-m}A,\ z_1A,\ z_2A,\dots,z_mA$ are generators for the range space $R(A)$. Since $z_iA=0$ $(i=1,2,\dots,m)$, vectors $x_1A,\ x_2A,\dots,x_{n-m}A$ are also generators for $R(A)$.

These vectors are linearly independent. If $c_1(x_1A)+c_2(x_2A)+\cdots+c_{n-m}(x_{n-m}A)=0$ for $c_i\in F$, then $(\sum_{i=1}^{n-m}c_ix_i)A=0$ and $\sum_{i=1}^{n-m}c_ix_i\in N(A)$. Since the set $\{x_1,\ x_2,\dots,x_{n-m},\ z_1,\ z_2,\dots,z_m\}$ is an independent set, this implies that c_i are all 0. Hence if we set $y_i=x_iA$ $(i=1,2,\dots,n\text{-}m)$ $\{y_i:\ i=1,2,\dots,n\text{-}m\}$ is a basis for $R(A)$. Let $[y_1,y_2,\dots,y_k]$ denote the subspace of the vector space $V(F)$ generated by the vectors y_i $(i=1,2,\dots,k)$. Then $R(A)=[y_1,y_2,\dots,y_{n-m}]$ and hence there is a set $\{y_{i_j}:\ j=1,2,\dots,k\}$ of vectors in the set $\{y_i:\ i=1,2,\dots,n\text{-}m\}$ such that $\{y_{i_j}:\ j=1,2,\dots,k\}$ is a basis for the space $R(A)\cap N(B)$. Without loss of generality, we may assume that $R(A)\cap N(B)=[y_1,y_2,\dots,y_k]$ and $k\leq n-m$. Then $\{y_iB:\ i=k+1,k+2,\dots,n-m\}$ is an independent set. To see this, assume that $c_{k+1}(y_{k+1}B)+c_{k+2}(y_{k+2}B)+\cdots+c_{n-m}(y_{n-m}B)=0$. Then $u=\sum_{i=k+1}^{n-m}c_iy_i$ $\in R(A)\cap N(B)$ and hence we have the following expression $u=c_1y_1+c_2y_2+\ldots+c_ky_k$ for some $c_i\in F$, $(i=1,2,\ldots,k)$. Since the set $\{y_i: i=1,2,\ldots,n-m\}$ is a linearly independent set, this implies that $c_i=0$ $(i=k+1,k+2,\ldots,n\text{-}m)$. Thus $\{y_iB$ $: i=k+1,\ k+2,\ldots,n\text{-}m\}$ are linearly independent vectors. Now we see that $\rho(AB)=\dim[y_1B,y_2B,\ldots,y_{n-m}B]=\dim[y_{k+1}B,y_{k+2}B,\ldots,y_{n-m}B]$ $=n-m-k=\rho(A)-\dim(R(A)\cap N(B))$. This proves the theorem.

3. We shall have a trivial application of Theorem 1 in semigroups. Let S be a semigroup. We define aLb $(a,b\in S)$ to mean that a and b generate the same principal l·ft ideal of S. In other words, L is the subset of $S\times S$ consisting of all pairs (a,b) such that $a\cup Sa=b\cup Sb$. It is not hard to see that L is an equivalence relation on S such that if aLb then $acLbc$ for all $c\in S$. If aLb, we say that a and b are L-equivalent. By L_a $(a\in S)$ we mean that the set of all elements of S which are L-equivalent to a. Dually we define aRb to mean that a and b generate the same principal right ideal of S. The join of the equivalence relations L and R is denoted by D. If $a\in R$ and $b\in L$, then aDb if and only if $R\cap L\neq\phi$, the empty set. We define $H=L\cap R$.

We now list some known properties [1, p. 59] of a semigroup $L(V)$ of all linear transformations of a vector space $V(F)$.

(3) $L(V)$ is a regular semigroup.

(4) Two elements of $L(V)$ are L-equivalent if and only if they have the same range space.

(5) Two elemlents of $L(V)$ are R-equivalent if and only if they have the same null space.

(6) Two elements of $L(V)$ are D-equivalent if and only if they have the same rank.

(7) Two elements of $L(V)$ are H-equivalent if and only if they have the same range space and the same null space.

We shall have a different proof of the following [1, Theorem2. 4].

THEOREM. *If a and b are two elements in* $L(V)$, *then* $L_aR_b=\{xy: x\in L_a$ *and* $y\in R_b\}\subset D_c$, *where* $c=ab$.

PROOF. If $x\in L_a$ and $y\in R_b$, then $R(x)=R(a)$ and $N(y)=N(b)$ by the above (4) and (5). It follows from Theorem 1 that $\rho(xy)=\rho(x)-\dim(R(x)\cap N(y))$

$=\rho(a)-\dim(R(a)\cap N(b))=\rho(ab)$. The theorem follows from the above (6).

We have the following.

COROLLARY. *If a and b are two elements in* $L(V)$, *then* $H_aH_b\subset D_c$, *where* $c=ab$.

We have in general that $R_aL_b\not\subset D_{ab}$.

4. REMARK. In the semigroup theory, one of the most important theorems is Green's Lemma [1, Lemma 2.2]. Using Green's Lemma, Miller and Clifford proved the following very important theorem.

THEOREM (Miller and Clifford). *If a and b are elements of a semigroup S, then* $ab\in R_a\cap L_b$ *if and only if* $R_b\cap L_a$ *contains an idempotent. If this is the case, then* $aH_b=H_ab=H_aH_b=R_a\cap L_b=H_{ab}$.

A blemish of the above theorem is that the theorem does not include the case when $R_b\cap L_a$ does not contain an idempotent. Therefore we set the following conjecture which is true if S is $L(V)$.

CONJECTURE. If a and b are elements of a semigroup S, and if $R_b\cap L_a$ does not contain an idempotent, then ${}_aH_b=\bigcup_{x\in H_b}H_{ax}$.

PROBLEM. Generalize Green's Lemma.

The author wishes to express his gratitute to Professor A.H. Clifford of Tulane University for his letter and the following proof of theorem 1.

The Proof of Theorem 1 by Professor A.H. Clifford:

Let $\overline{B}=B|R(A)$, the linear transformation B restricted to $R(A)$. $R(\overline{B})=R(AB)$, so $\rho(AB)=\rho(\overline{B})$. But $\rho(\overline{B})=\dim R(A)-\dim N(\overline{B})$, $\dim R(A)=\rho(A)$, and $N(\overline{B})=R(A)\cap N(B)$. Putting then together gives the desired result.

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REFERENCES

[1]. A.H. Clifford and G.B. Preston: *The algebraic theory of semigroups*, Mathematical Surveys of the Amer. Math. Soc., Vol. 7-1, Providence, R.I., 1961.

[2]. N.Jacobson: *Lectures in abstract algebra, Vol. II -Linear Algebra*, D.Van Nostrand Company, Inc., Princeton, New Jersey, 1953.

[3]. J.B.Kim ; *On singular matrices*, Czechoslovak Math. Journal, (1968), 274-277.

[4]. J.B.Kim ; *On singular matrices*, Notices of Amer. Math. Soc., (1967), 663.