ON THE COMPLETENESS OF A SYMMETRIC GENERALIZED UNIFORM SPACE

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In this paper the concept of completeness is defined for a symmetric generalized uniform space (introduced by the author in [4]). A number of theorems are proved to indicate that the definition is a proper one.

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In this paper every space will be a symmetric generalized uniform space unless otherwise indicated.

DEFINITION 1. $(X, \mathscr{U})$ is *totally bounded* iff for every V in $\mathscr{U}$ there exists $x_1, \cdots, x_n$ in X such that $X=V(x_1) \cup \cdots \cup V(x_n)$.

DEFINITION 2. A filter $\mathscr{F}$ in X is *weakly Cauchy* iff for every U in $\mathscr{U}$ there exists x in X such that $U[x] \in \mathscr{F}$.

DEFINITION 3. $(X, \mathscr{U})$ is *complete* iff every weakly Cauchy filter in X has a cluster point in X.

DEFINITION 4. $(X, \mathscr{U})$ is *Δ-complete* iff whenever $(X, \mathscr{U})$ is uniformly isomorphic to a dense subspace $(X_1, \mathscr{U}_1)$ of $(X_2, \mathscr{U}_2)$, $X_1=X_2$.

DEFINITION 5. A *correct uniform space* (c. f. [1]) is a separated, symmetric generalized uniform space that has the additional property that for any U in $\mathscr{U}$ there exists a V in $\mathscr{U}$ such that $V \circ V \subseteq U$.

THEOREM 1. *If $(X, \mathscr{U})$ is a symmetric uniform space, then a filter $\mathscr{F}$ in X is Cauchy iff it is weakly Cauchy.*

The proof is straightfoward.

THEOREM 2. *If $(X, \mathscr{T})$ is a symmetric, connnected topological space, then there exists a totally bounded, symmetric generalized uniformity $\mathscr{U}$ on X such that $\mathscr{T}(\mathscr{U})=\mathscr{T}$, and every filter in X is weakly Cauchy.*

PROOF. Let $\mathscr{P}$ be a symmetric generalized proximity on X such that $\mathscr{T}(\mathscr{P})=\mathscr{T}$. Let $U_{A,B}=(X\times X)-((A\times B)\cup(B\times A))$. Let $\mathscr{B}=\{U_{A,B}|(A, B)\notin\mathscr{P}\}$. By the lemma on page 5 in [4] $\mathscr{B}$ is a base for a symmetric generalized uniformity $\mathscr{U}$ on X such that $\mathscr{U}\in\Pi(\mathscr{P})$. It is easy to show that $\mathscr{U}$ is totally bounded. We note that $U_{\bar{A},\bar{B}}\subseteq U_{A,B}$ for all A, B such that $(A, B)\notin\mathscr{P}$. But since $\mathscr{T}$ is connected, there exists $x_0\in X-(\bar{A}\cup\bar{B})$. Consequently, for every U in $\mathscr{U}$ there exists x in X such that $U[x]=X$; hence every filter in X is weakly Cauchy.

THEOREM 3. *If $(X, \mathscr{U})$ is totally bounded then every ultrafilter in X is a weakly Cauchy filter.*

PROOF. Fix V in $\mathscr{U}$. There exists $x_1, \cdots, x_n$ in X such that X is equal to $V(x_1)\cup\cdots\cup V(x_n)$. But since $X\in\mathscr{F}$, it is easily shown (c.f. last line on page 221 of [2]) that for some m $(1\leq m\leq n)$ $V(x_m)\in\mathscr{F}$; hence $\mathscr{F}$ is weakly Cauchy.

THEOREM 4. *$(X, \mathscr{U})$ is compact iff it is complete and totally bounded.*

PROOF. If $(X, \mathscr{U})$ is compact, then it is an immediate consequence of theorem 4 in [4] that $(X, \mathscr{U})$ is totally bounded. Let $\mathscr{F}$ be any weakly Cauchy filter in X. Sincce $(X, \mathscr{U})$ is compact, $\mathscr{F}$ has a cluster point; hence $(X, \mathscr{U})$ is complete. Conversely, let $\mathscr{F}$ be an ultrafilter in X. Since $(X, \mathscr{U})$ is totally bounded, $\mathscr{F}$ is weakly Cauchy; and since $(X, \mathscr{U})$ is complete $\mathscr{F}$ has a cluster point. But an ultrafilter converges to each of its cluster points; consequently, $(X, \mathscr{U})$ is compact.

COROLLARY 1. *Let $(X, \mathscr{U})$ be a correct uniform space. Then $(X, \mathscr{U})$ is compact iff it is totally bounded and every infrafilter in X is a neighborhood filter.*

PROOF. This is an immediate consequence of theorem 1, theorem 4 and the fact (proved in [1]) that $(X, \mathscr{U})$ is complete iff every infrafilter in X is a neighborhood filter.

THEOREM 5. *$(X, \mathscr{U})$ is totally bounded iff every filter in X is contained in a weakly Cauchy filter.*

PROOF. Let $\mathscr{F}$ be a filter. $\mathscr{F}$ is contained in some ultrafilter $\mathscr{F}_1$ which by theorem 3 is weakly Cauchy. For a nice proof of the converse see page 192 in [7].

THEOREM 6. *If* $(X, \mathscr{U})$ *is complete, then every closed and totally bounded subspace is compact.*

PROOF. By theorem 4 it is sufficient to show that every closed subspace of a complete space is complete. The proof of that fact is straightfoward.

THEOREM 7. *If* $(X, \mathscr{U})$ *is a totally bounded, dense subspace of* $(X_1, \mathscr{U}_1)$, *and if every element of every weakly Cauchy filter in* X_1 *has a non-void interior (relative to* $\mathscr{T}(\mathscr{U}_1)$), *and if every weakly Cauchy filter (relative to* $\mathscr{U}$) *in* X *has a cluster point in* X_1, *then* $(X_1, \mathscr{U}_1)$ *is complete.*

PROOF. Let $\mathscr{F}$ be weakly Cauchy in X_1 such that for every F in $\mathscr{F}$ $F^0 \neq \phi$. Since X is dense in X_1, $F \cap X \neq \phi$ for every F in $\mathscr{F}$. Let $\mathscr{B}=\{F \cap X \mid F \text{ in } \mathscr{F}\}$. Clearly, $\mathscr{B}$ is a base for a filter $\mathscr{F}$ in X which by theorem 5 is contained in a weakly Cauchy filter $\mathscr{F}_1$ in X. But there exists an x_0 in X such that x_0 is a cluster point of $\mathscr{F}_1$. Fix U in $\mathscr{U}$. Let $F \in \mathscr{F}$. Then $U[x_0] \cap (F \cap X) \neq \phi$; so that $U[x_0] \cap F \neq \phi$. Hence x_0 is a cluster point of $\mathscr{F}$.

THEOREM 8. *If* $(X, \mathscr{U})$ *is separated, and is* Δ*-complete, then every weakly Cauchy filter in* X *is a neighborhood filter.*

PROOF. Suppose there exists at least one weakly Cauchy filter in X which is not a neighborhood filter. Let X_2 be the family of all weakly Cauchy filters in X. Let X_1 be the family of all neighborhood filters in X. To construct the uniformity on X_2 in the proper way we assign to each filter P in the set X_2 a point x_p in X in the following way: $x_p=x_1$ if $\mathscr{N}(x_1)=P$, and x_p is any point in X if $\mathscr{N}(x) \neq P$ for every x in X. For every U in $\mathscr{U}$ let $\bar{U}$ be equal to $\{(P_1, P_2) \mid (x_{p_1}, x_{p_2}) \in U\}$. Let $\mathscr{U}_\beta=\{\bar{U} \mid U \text{ in } \mathscr{U}\}$. It is easily shown that $\mathscr{U}_\beta$ is a base for a symmetric generalized uniformity $\mathscr{U}_2$ on X_2 such that if $\mathscr{U}_1$ is the relativization of $\mathscr{U}_2$ to X_1, then $(X, \mathscr{U})$ is uniformly isomorphic to $(X_1, \mathscr{U}_1)$ and X_1 is dense in X_2 (c.f. [3] page 297).

THEOREM 9. *Suppose* $(X, \mathscr{U})$ *is separated. If* $(X_2, \mathscr{U}_2)$ *(as constructed in the proof of theorem* 8) *is complete, then* $(X, \mathscr{U})$ *is complete.*

PROOF. Let $\mathscr{F}$ be weakly Cauchy in X. Let $\mathscr{F}^*$ be the natural image of

$\mathscr{F}$ in X_1. $\mathscr{F}^*$ is a base for a filter $\mathscr{F}_1{}^*$ in X_2. Clearly, $\mathscr{F}_1{}^*$ is weakly Cauchy with respect to $\mathscr{U}_2$. Thus $\mathscr{F}_1{}^*$ has a cluster point P_1 in X_2. Clearly, P_1 is a cluster point for $\mathscr{F}^*$. Let $F \in \mathscr{F}$, and let F^* be the natural image of F in X_1. Fix $\bar{U}$ in $\mathscr{U}_\beta$. There exists $\mathscr{N}(x_1) \in \bar{U}[P_1] \cap F^*$; so that $x_1 \in U[x_{p_1}] \cap F$; consequently, x_{p_1} is a cluster point of $\mathscr{F}$ and $(X, \mathscr{U})$ is complete.

THEOREM 10. *A correct space is complete iff it is Δ-complete.*

The proof is straightfoward.

The following theorem is a generalization of the theorem of Niemytzki and Tychonoff (c.f. [6]) namely that a metric space is compact iff it is complete in every metric. Another generalization of this theorem is obtained in [8].

THEOREM 11. *A symmetric topological space $(X, \mathscr{T})$ is compact iff it is complete with respect to every compatible symmetric generalized uniformity on X.*

PROOF. This is an immediate consequence of the lemma on page 5 in [4] and theorem 4 above.

REMARK: Note that in this paper the definition of completeness for a symmetric generalized uniform space is the same as that for a quasi-uniform space (as defined in [5]) and is equivalent to that for a correct space (as defined in [1]). Theorem 4 is proved in [5] in essentially the same way that it is proved here. Also, for a discussion of the history of the weakly Cauchy filter concept see [8].

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