ON PROPERTY OF BESSEL TRANSFORM

By M. L. Maheshwari

1. Introduction

We shall denote

$$g(y)=\int_0^\infty f(x)K_\gamma(xy)(xy)^{\frac{1}{2}}dx=M^\gamma[f(x)], \tag{1.1}$$

the Bessel transform of order γ of $f(x)$, where y may be a real or complex variable. This transformation was introduced by Meijer [1]. It was further investigated by Erdélyi [2] and Boas [4].

Let $\gamma=\pm\frac{1}{2}$, (1.1) reduces to the Laplace transform,

$$\phi(p)=p\int_0^\infty e^{-pt}f(t)dt,\ \text{Re}\ p>0,$$

then $\phi(p)$ is said to be the image of $f(t)$ and $f(t)$ the original of $\phi(p)$, and is denoted symbolically by

$$f(t)\phi\doteqdot(p)\ \text{or}\ \phi(p)\doteqdot f(t).$$

The object of this paper is to establish a new property of this transformation and a generalized result as its application.

2. THEOREM. *Let* (i) $M^\gamma[f(x)]=g(y)$,

$$\text{(ii)}\ M^\gamma\left[x^{-2}g\left(\frac{1}{x}\right)\right]=\pi\phi_1(y), \tag{2.1}$$

then

$$M^{2\gamma}\left[x^{-\frac{1}{2}}f\left(\frac{x^2}{4}\right)\right]=y^{-\frac{1}{2}}\phi_1(y^2), \tag{2.2}$$

provided

$$x^{\left(\pm\frac{1}{2}\pm\gamma\right)}f(x),\ x^{\left(-\frac{3}{2}\pm\gamma\right)}g\left(\frac{1}{x}\right)=o(x^\alpha),$$

Re $\alpha>-1$ *for small* x *and* $f(x)$, $g\left(\frac{1}{x}\right)$ *are bounded and absolutely integrable in* $(0,\infty)$.

Further, let

$$M^{2\gamma}\left[x^{\frac{1}{2}}\phi_1\left(\frac{1}{x^2}\right)\right]=2^{1/2}\pi\phi_2(y) \tag{2.3}$$

$$M^{2^2\gamma}\left[x^{-\frac{3}{2}}\phi_2\left(\frac{1}{2x^2}\right)\right]=\frac{\pi}{2^{1/2}}\phi_3(y), \tag{2.4}$$

$$M^{2^3\gamma}\left[x^{-\frac{3}{2}}\phi_3\left(\frac{1}{2x^2}\right)\right]=\frac{\pi}{2^{5/2}}\phi_4(y), \tag{2.5}$$

....

$$M^{2^{n-1}\gamma}\left[x^{-\frac{3}{2}}\phi_{n-1}\left(\frac{1}{2x^2}\right)\right]=\frac{\pi\phi_n(y)}{2^{(2^{n-2}-3/2)}}. \tag{2.6}$$

Then

$$M^{2^n\gamma}\left[x^{(2^{n-1}-3/2)}f\left(\frac{x^{2^n}}{2^{2^n}}\right)\right]=y^{\frac{3}{2}}\phi_n\left(\frac{y^2}{2}\right) \tag{2.7}$$

under the conditions mentioned above and Re $\gamma>-\frac{1}{2^n}$, Re $n>1$ *and* n *is an integer.*

PROOF. Let $\int_0^\infty f(x)K_\gamma(xy)(xy)^{\frac{1}{2}}dx=g(y)$.

Multiplying both sides by $y^{-\frac{1}{2}}K_\gamma\left(\frac{a}{y}\right)$ and integrating with respect to y between the limits 0 to ∞, we obtain

$$\int_0^\infty y^{-1/2}K_\gamma\left(\frac{a}{y}\right)dy\int_0^\infty f(x)K_\gamma(xy)(xy)^{\frac{1}{2}}dx=\int_0^\infty y^{-\frac{1}{2}}K_\gamma\left(\frac{a}{y}\right)g(y)dy$$

On changing the order of integrations, which is permissible by the conditions given in the theorem and evaluating the y-integral on both hand side and using (2.1) on R.H.S., we obtain

$$a^{\frac{1}{2}}\int^\infty x^{-\frac{1}{2}}f(x)K_{2\gamma}[2(ax)^{\frac{1}{2}}]dx=\phi_1(a).$$

Hence, we obtain (2.2).

We obtain $x^{\frac{1}{2}}\phi_1\left(\frac{1}{x^2}\right)$ from (2.2), substituting it, we have from (2.3)

$$2^{\frac{1}{2}}\pi\phi_2(y)=\int_0^\infty K_{2\gamma}(xy)(xy)^{\frac{1}{2}}\left[x^{-\frac{1}{2}}\int_0^\infty f\left(\frac{t^2}{4}\right)K_{2\gamma}\left(\frac{t}{x}\right)dt\right]dx.$$

On changing the order of integrations and evaluating the x-integral as before, we obtain

$$M^{2^2\gamma}\left[t^{\frac{1}{2}}f\left(\frac{t^4}{16}\right)\right]=y^{\frac{3}{2}}\phi_2\left(\frac{y^2}{2}\right) \tag{2.8}$$

Substituting for $x^{-\frac{3}{2}}\phi_2\left(\frac{1}{2x^2}\right)$, we obtain from (2.4)

$$2^{-\frac{1}{2}}\pi\phi_3(y)=\int_0^\infty K_{2^2\gamma}(xy)(xy)^{\frac{1}{2}}\left[x^{-\frac{1}{2}}\int_0^\infty t f\left(\frac{t^4}{2^4}\right)K_{2^2\gamma}\left(\frac{t}{x}\right)dt\right]dx.$$

Proceeding as before, we obtain

$$M^{2^3\gamma}\left[t^{\frac{5}{2}}f\left(\frac{t^8}{2^8}\right)\right]=y^{\frac{3}{2}}\phi_3\left(\frac{y^2}{2}\right) \tag{2.9}$$

Proceeding successively, we have the result (2.7).

Let $$M^{2^n\gamma}\left[\frac{1}{x^{3/2}}\phi_n\left(\frac{1}{2x^2}\right)\right]=\frac{\pi\phi_{n+1}(y)}{2^{(2^{n-1}-3/2)}}. \tag{2.10}$$

We obtain $x^{-\frac{3}{2}}\phi_n\left(\frac{1}{2x^2}\right)$ from (2.7), substituting it we obtain from (2.10)

$$2^{\left(\frac{3}{2}-2^{n-1}\right)}\pi\phi_{n+1}(y)=\int_0^\infty K_{2^n\gamma}(xy)(xy)^{\frac{1}{2}}$$

$$\times\left[x^{-\frac{1}{2}}\int_0^\infty t^{(2^{n-1}-1)}f\left(\frac{t^{2^n}}{2^{2^n}}\right)K_{2^n\gamma}\left(\frac{t}{x}\right)dt\right]dx.$$

On changing the order of integrations and evaluating the x-integral as before, we obtain

$$M^{2^{n+1}\gamma}\left[t^{\left(2^n-\frac{3}{2}\right)}f\left(\frac{t^{2^{n+1}}}{2^{2^{n+1}}}\right)\right]=y^{\frac{3}{2}}\phi_{n+1}\left(\frac{y^2}{2}\right).$$

We thus find that if (2.7) is true for n, it is also true for $(n+1)$, i.e., for the next higher order. But we have seen that it is true for $n=2$ and therefore it is true for $n=3$. Since it is true for $n=3$, so it is true for $n=4$ and so on. Hence (2.7) is true for all positive integral values of n except one.

COROLLARY. *Let* $\gamma=\frac{1}{2^{n+1}}$. *We obtain from* (2.7)

$x^{\left(2^{n-1}-\frac{3}{2}\right)}f\left(\frac{x^{2^n}}{2^{2^n}}\right)\doteqdot\left(\frac{2}{\pi}\right)^{\frac{1}{2}}p^{\frac{5}{2}}\phi_n\left(\frac{p^2}{2}\right)$, *under the conditions mentioned in the theorem.*

3. Application.

Let $f(x)=x^{\left(\mu-\frac{3}{2}\right)}{}_pF_q\left[\begin{matrix}\alpha_1,\cdots\cdots,\alpha_p; \\ \beta_1,\cdots\cdots,\beta_q;\end{matrix}\ -\delta x^2\right]$,

$$g(y)=2^{(\mu-2)}\Gamma\left(\frac{\mu\pm\gamma}{2}\right)^{*}y^{\left(\frac{1}{2}-\mu\right)}{}_{p+2}F_q\left[\begin{matrix}\alpha_1,\cdots\cdots,\alpha_p,\frac{\mu\pm\gamma}{2}; \\ \beta_1,\cdots\cdots\cdots\cdots,\beta_q;\end{matrix}\ -\frac{4\delta}{y^2}\right],$$

$p\leq q-1$, Re $y>0$, and Re $(\mu\pm\gamma)>0$.

$$\therefore\ \phi_1(y)=\frac{2^{(2\mu-5)}}{\pi y^{\left(\mu-\frac{3}{2}\right)}}\Gamma\left(\frac{\mu\pm\gamma}{2}\right)\Gamma\left(\frac{\mu\pm\gamma-1}{2}\right)$$

$$\times{}_{p+4}F_p\left[\begin{matrix}\alpha_1,\cdots\cdots\alpha_p,\frac{\mu\pm\gamma}{2},\ \frac{\mu\pm\gamma-1}{2}; \\ \beta_1,\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\beta_q;\end{matrix}\ -\frac{16\delta}{y^2}\right]\qquad [3,\ p.153],$$

$p\leq q-3$, Re$(\mu\pm\gamma)>1$ and Re $y>0$.

Hence we obtain from (2.2)

$$M^{2\gamma}\left[x^{\left(2\mu-\frac{7}{2}\right)}{}_pF_q\left\{\begin{matrix}\alpha_1,\cdots\cdots,\alpha_p; \\ \beta_1,\cdots\cdots,\beta_q;\end{matrix}\ -\frac{\delta x^4}{16}\right\}\right]=\frac{2^{(4\mu-8)}}{\pi y^{\left(2\mu-\frac{5}{2}\right)}}$$

$$\times\Gamma\left(\frac{\mu\pm\gamma}{2}\right)\Gamma\left(\frac{\mu\pm\gamma-1}{2}\right){}_{p+4}F_q\left\{\begin{matrix}\alpha_1,\cdots\cdots,\alpha_p,\frac{\mu\pm\gamma}{2},\ \frac{\mu\pm\gamma-1}{2}; \\ \beta_1,\cdots\cdots\cdots\cdots\cdots\cdots,\beta_q;\end{matrix}\ -\frac{16\delta}{y^4}\right\},\qquad (3.1)$$

Re$(\mu\pm y)>1$ if $p<q-3$,
Re $y>0$ if $p+4\leq q$,

and Re$\left\{y+4\delta\exp\left(\frac{\pi ri}{2}\right)\right\}>0$ for $r=0,1,2,3$, if $p=q-3$.

We obtain $\phi_2(y)$ from (2.3), on using (3.1). Let $n=2$. We obtain from (2.7)

$$M^{4\gamma}\left[x^{\left(4\mu-\frac{11}{2}\right)}{}_pF_q\left\{\begin{matrix}\alpha_1,\cdots\cdots,\alpha_p; \\ \beta_1,\cdots\cdots,\beta_q;\end{matrix}\ -\delta\frac{x^8}{2^8}\right\}\right]$$

$$=\frac{2^{(12\mu-19)}}{\pi^3y^{\left(4\mu-\frac{9}{2}\right)}}\Gamma\left(\frac{\mu\pm\gamma}{2}\right)\Gamma\left(\frac{\mu\pm\gamma-1}{2}\right)\Gamma\left(\frac{\mu\pm\gamma}{2}\pm\frac{1}{4}\right)$$

$$\times{}_{p+8}F_q\left\{\begin{matrix}\alpha_1,\cdots\cdots\alpha_p,\frac{\mu\pm\gamma}{2},\ \frac{\mu\pm\gamma-1}{2},\ \frac{\mu\pm\gamma}{2}\pm\frac{1}{4}; \\ \beta_1,\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\beta_q;\end{matrix}\ -\frac{2^{16}\delta}{y^8}\right\},\qquad (3.2)$$

Re $y>0$ if $p+8\leq q$, Re $(\mu\pm y)>1$ if $p<q-7$,

Re $\left\{y+2^8\delta\exp\left(\frac{\pi ri}{2}\right)\right\}>0$, for $r=0,1,2,\cdots,7$ if $p=q-7$.

Similarly, we get $\phi_3(y)$ from (2.4), on using (3.2). Let $n=3$, we get from (2.7)

---

* $\left(\frac{\mu\pm\gamma}{2}\right)=\Gamma\left(\frac{\mu+\gamma}{2}\right)\Gamma\left(\frac{\mu-\gamma}{2}\right)$

and $\Gamma\left(\frac{\mu\pm\gamma}{2}\pm\frac{1}{4}\right)=\Gamma\left(\frac{\mu+\gamma}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\mu+\gamma}{2}-\frac{1}{4}\right)\Gamma\left(\frac{\mu-\gamma}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\mu-\gamma}{2}-\frac{1}{4}\right)$

$$M^{8\gamma}\Big[x^{\left(8\mu-\frac{19}{2}\right)}{}_pF_q\Big\{\begin{matrix}\alpha_1,\cdots\cdots\alpha_p;\\ \beta_1,\cdots\cdots,\beta_q:\end{matrix}-\delta\frac{x^{16}}{2^{16}}\Big\}\Big]$$

$$=\frac{2^{(32\mu-44)}}{\pi^7 y^{\left(8\mu-\frac{17}{2}\right)}}\Gamma\Big(\frac{\mu\pm\gamma}{2}\Big)\Gamma\Big(\frac{\mu\pm\gamma-1}{2}\Big)\Gamma\Big(\frac{\mu\pm\gamma}{2}\pm\frac{1,2,3}{8}\Big)^*$$

$$\times{}_{p+16}F_q\Big\{\begin{matrix}\alpha_1,\cdots\cdots\alpha_p,\ \frac{\mu\pm\gamma}{2},\ \frac{\mu\pm\gamma-1}{2},\ \frac{\mu\pm\gamma}{2}\pm\frac{1,2,3}{8};\\ \beta_1,\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\ \beta_q;\end{matrix}\ -\frac{2^{48}\delta}{y^{16}}\Big\},\qquad(3.3)$$

Re $y>0$ if $p+16\le q$, Re $\left\{y+2^{16}\delta\exp\left(\frac{\pi ri}{2}\right)\right\}>0$ for $r=0,1,2,\cdots\cdots,$ 15 if p $=q-15$, and Re $(\mu\pm\gamma)>1$ if $p<q-15$.

Proceeding successively we obtain

$$M^{(N\gamma)}\Big[x^{\left(N\mu-N-\frac{3}{2}\right)}{}_pF_q\Big\{\begin{matrix}\alpha_1,\cdots\cdots,\alpha_p;\\ \beta_1,\cdots\cdots,\beta_q;\end{matrix}-\delta\Big(\frac{x}{2}\Big)^{2N}\Big\}\Big]$$

$$=\frac{2^{\{(n+1)N\mu-(n+2)N-(n+1)\}}}{\pi^{(N-1)}y^{\{(\mu-1)N-1/2\}}}\Gamma\Big(\frac{\mu\pm\gamma}{2}\Big)\Gamma\Big(\frac{\mu\pm\gamma}{2}-\frac{1}{2}\Big)\Gamma\Big(\frac{\mu\pm\gamma}{2}\pm\frac{1,2,3,\cdots,\left(\frac{N}{2}-1\right)}{N}\Big)$$

$$\times{}_{p+2N}F_q\Big[\begin{matrix}\alpha_1,\cdots\cdots,\alpha_p,\ \frac{\mu\pm\gamma}{2},\ \frac{\mu\pm\gamma-1}{2},\ \frac{\mu\pm\gamma}{2}\pm\frac{1,2,3,\cdots,\left(\frac{N}{2}-1\right)}{N};\\ \beta_1,\cdots\cdots,\beta_q;\end{matrix}-\delta\Big(\frac{N}{y}\Big)^{2N}\Big]\qquad(3.4)$$

where $N=2^n$, Re $n>1$ and n is an integer. Re$(\mu\pm y)>1$, Re $y>0$ if $p+2N\le q$, Re$\left\{y+2^{2N}\delta\exp\left(\frac{\pi ri}{2}\right)\right\}>0$ for $r=0,1,\cdots,(2N-1)$ if $p=q-(2N-1)$

Equation (3.4) can be proved by Mathematical Induction as the theorem.

Particular cases:

(i) Let $n=2$, $(N=4)$, $p=0$, $q=8$, and

$$\beta_1,\ \beta_2,\ \cdots,\ \beta_8=\frac{\mu\pm\gamma}{2},\ \frac{\mu\pm\gamma}{2}-\frac{1}{2},\ \frac{\mu\pm\gamma}{2}\pm\frac{1}{4}.$$

We obtain from (3.4)

$$M^{4\gamma}\Big[x^{\left(4\mu-\frac{11}{2}\right)}{}_0F_8\Big\{\frac{\mu\pm\gamma}{2},\ \frac{\mu\pm\gamma}{2}-\frac{1}{2},\ \frac{\mu\pm\gamma}{2}\pm\frac{1}{4};-\delta\Big(\frac{x}{2}\Big)^8\Big\}\Big]$$

---

* $\Gamma\left(\frac{\mu\pm\gamma}{2}\pm\frac{1,2,3}{8}\right)=\Gamma\left(\frac{\mu\pm\gamma}{2}\pm\frac{1}{8}\right)\Gamma\left(\frac{\mu\pm\gamma}{2}\pm\frac{1}{4}\right)\Gamma\left(\frac{\mu\pm\gamma}{2}\pm\frac{3}{8}\right)$

$$=\frac{2^{(12\mu-19)}}{\pi^3 y^{\left(4\mu-\frac{9}{2}\right)}}\Gamma\left(\frac{\mu\pm\gamma}{2}\right)\Gamma\left(\frac{\mu\pm\gamma-1}{2}\right)\Gamma\left(\frac{\mu\pm\gamma}{2}\right)\pm\frac{1}{4}\Big)\exp\left\{-\delta\left(\frac{4}{y}\right)^8\right\},\quad (3.5)$$

$\mathrm{Re}(\mu\pm\gamma)>1.$

Let $\gamma=\frac{1}{8}$, we obtain from (3.5)

$$x^{\left(4\mu-\frac{11}{2}\right)}{}_0F_8\left\{\frac{\mu}{2}\pm\frac{1,3,5,7,9}{16};\ -\delta\left(\frac{x}{2}\right)^8\right\}$$

$$\doteqdot p^{\left(\frac{11}{2}-4\mu\right)}\exp\left\{-\delta\left(\frac{4}{p}\right)^8\right\}.$$

(ii) Let $p=3,\ q=8,\ \delta=108,\ n=2,\ (N=4)$,

$\alpha_1,\ \alpha_2,\ \alpha_3=\frac{1}{3}(\rho_1+\rho_2-1),\ \frac{1}{3}(\rho_1+\rho_2),\ \frac{1}{3}(\rho_1+\rho_2+1)$, and

$\beta_1,\ \cdots\cdots,\ \beta_8=\rho_1,\ \rho_2,\ \frac{1}{2}\rho_1,\ \frac{1}{2}\rho_2,\ \frac{1}{2}(\rho_1+1),\ \frac{1}{2}(\rho_2+1),\ \frac{1}{2}(\rho_1+\rho_2-1),$

$\frac{1}{2}(\rho_1+\rho_2)$.

We obtain from (3.4)

$$M^{4\gamma}\left[x^{\left(4\mu-\frac{11}{2}\right)}{}_0F_2(\rho_1,\rho_2;x^4)\,{}_0F_2(\rho_1,\rho_2;-x^4)\right]$$

$$=\frac{2^{(12\mu-19)}}{\pi^3 y^{\left(4\mu-\frac{9}{2}\right)}}\Gamma\left(\frac{\mu\pm\gamma}{2}\right)\Gamma\left(\frac{\mu\pm\gamma-1}{2}\right)\Gamma\left(\frac{\mu\pm\gamma}{2}\pm\frac{1}{4}\right)$$

$$\times{}_{11}F_8\left[\begin{matrix}\frac{1}{3}(\rho_1+\rho_2-1),\frac{1}{3}(\rho_1+\rho_2),\frac{1}{3}(\rho_1+\rho_2+1),\frac{\mu\pm\gamma}{2},\frac{\mu\pm\gamma-1}{2},\frac{\mu\pm\gamma}{2}\pm\frac{1}{4};\\ \rho_1,\rho_2,\frac{1}{2}\rho_1,\frac{1}{2}\rho_2,\frac{1}{2}(\rho_1+1),\frac{1}{2}(\rho_2+1),\frac{1}{2}(\rho_1+\rho_2-1),\frac{1}{2}(\rho_1+\rho_2);\end{matrix}\ -\frac{3^3 4^9}{y^8}\right],$$

provided one of the parameters in the numerator is a negative integer and Re $(\mu\pm\gamma)>1$.

In the end I wish to record my respectful thanks to Dr. S. C. Mitra, Emeritus Research Professor in Mathematics, B. I. T. S., Pilani, for his help and guidance in the preparation of this paper.

Birla Institute of Technology and Science
Pilani (Rajasthan), India

REFERENCES

[1] Meijer, C.S., 1940; Proc. Amsterdam Akad. Wet. 43, 599—608 and 702—711.

[2] Erdelyi, Arthur, 1950—51; Rend. Sem. Mat. Univ. Torino 10, 217—234.

[3] Bateman Project; *Tables of integral transforms, V.*2, McGraw-Hill (1954).

[4] Boas, R.P., 1942; Bull. Amer. Math. Soc., 48, 286—294.