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Iterative Solutions of the Dirichlet Problem for $\triangle u = bu^{2n}$

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1. Introduction. We construct the solution of

$$\triangle u = bu^{2n} \tag{1}$$

in a given region, the value of u on the boundary of the region being known. The given boundary values of u and the constant b are nonnegative. The existence of the solution is demonstrated by the convergence of the iterative methods discussed.

The notation and terminology of this paper are based on [1]. Under the same conditions of [1], we can easily show that the solution u is unique by an appeal to the maximum principle and its corollary in [1]. Three iterative methods belonging to the class of linear iterative methods

$$\triangle u_{i+1} = bu_i^n[cu_i^n + (1-c)u_{i+1}^n]$$

are investigated.

2. Newton's Methods. The problem of this paper is to determine the nonnegative solution u of the integral equation

$$u(s) + b\int G(s, t)u^{2n}(t)dt - h(s) = 0 \tag{2}$$

which is equivalent to (1). Identifying $P(u)$ with the left-hand side of (2), one has

$$P'(u)v(s) = v(s) + 2nb\int G(s,t)u^{2n-1}v(t)dt,$$

and

$$P''(u)vw(s) = 2n(2n-1)b\int G(s, t)u^{2n-2}v(t)w(t)dt.$$

The initial function u_0 is to be an approximate solution of (2). If b is small, one may take $u_0 = h$.

From (2), we have

$$P(u_0)=b\int G(s,\ t)h^{2n}(t)dt.$$

Since the maximum value of h is unity, the maximum of f,

$$P(u_0)\leqq b\int G(s,\ t)\mathrm{dt}\leqq bG\ [1].$$

From the definition of the Fréchet derivative,

$$P'(u_0)u=u(s)+2nb\int G(s,\ t)h^{2n-1}(t)u(t)dt.$$

Using the bound of the Green's function [1],

$$\max\ 2nb\int G(s,\ t)h^{2n-1}(t)u(t)dt\leqq 2nbG_1 \max u(s).$$

If $2nbG_1<1$, the inverse of the operator $P'(u)$ exists and its norm is no greater than

$$B=1/(1-2nbG_1)\ [3].$$

We obtain the following inequality

$$\|[P'(u_0)]^{-1}P(u_0)\|\leqq\|[P'(u_0)]^{-1}\|\cdot\|P(u_0)\|\leqq B\cdot bG_1=\eta.$$

A bound for $P''(u)$ with u in a certain neighborhood of u_0 is needed. However, $P''(u)$ has the general bound

$$K=2n(2n-1)bG_1.$$

Finally, we need

$$\frac{1}{2}\geqq BK\,\eta=\frac{2n(2n-1)b^2G_1^2}{(1-2nbG_1)^2}. \tag{3}$$

If b is sufficiently small, this inequality will be satisfied. Thus for the present problem the Kantorovich Theorem [1] asserts that if b is so small that inequality (3) is satisfied, then the sequence u_i of solutions to

$$u_0(s)=h(s)$$

$$u_{i+1}(s)+2nb\int G(s,t)u_i^{2n-1}(t)u_{i+1}(t)dt$$

$$=h(s)+(2n-1)b\int G(s,t)u_i^{2n}(t)dt \qquad (i=0,1,2,\cdots\cdots)$$

converges to the solution u of (2), that solution being unique.

3. The Natural iteration. The successive iterates u_i are solutions of

$$u_0=h$$

$$\Delta u_{i+1}=bu_i^{2n} \qquad \text{in } R$$

$$u_i = f \qquad \text{on } S (i=0, 1, 2, \cdots\cdots)$$

or equivalently

$$u_{i+1}(s) = h(s) - b\int G(s, t) u_i^{2n}(t) dt.$$

h is positive in R [1]. Since $\int G(s, t) u_0^{2n}(t) dt$ is positive and zero on S, for small enough b, $u_1(s)$ will also be nonnegative in $R: 0 \leqq u_1$.

The solution u is no greater than h. For $\Delta(u-h) = bu^{2n} \geqq 0$, so that, by the maximum principle, $u-h \leqq 0$.

If b is so small that $0 \leqq u_1$, then

$$0 \leqq u_1 \leqq u_3 \leqq \cdots\cdots \leqq u_{2i+1} \leqq \cdots\cdots \leqq u \leqq \cdots\cdots \leqq u_{2i} \leqq \cdots\cdots \leqq u_2 \leqq u_0$$

by further application of the maximum principle.

The convergence of the sequence u_i is established as follows:

$$|u_{n+1} - u_n| = b|\int G(s, t) [u_n^{2n} - u_{n-1}^{2n}] (t) dt$$

$$= b|\int G(s, t)(u_n^n + u_{n-1}^n)(u_n^n - u_{n-1}^n)(t) dt$$

$$\leqq b \max |u_n^n - u_{n-1}^n| \cdot |\int G(s, t) \cdot (u_n^n + u_{n-1}^n) dt|$$

$$\leqq 2b \max |u_n^n - u_{n-1}^n| \cdot |\int G dt|$$

$$\leqq 2bG_1 \max |u_n^n - u_{n-1}^n|$$

$$\leqq 2bG_1 \max |u_n^{n-1} + u^{-2}{}_n{}^n u_{n-1} + \cdots\cdots + u_n^{n-1}| \cdot |u_n - u_n^n - 1|$$

$$\leqq 2nbG_1 \cdot \max |u_n - u_{n-1}|.$$

If $2nbG_1 < 1$, the sequence converges absolutely and uniformly.

4. The iteration for arbitrary positive values of b. For this method the sequence of functions u_i is taken to be:

$$u_0 = h$$

$$\Delta u_{i+1} = bu_i^n u_{i+1}^n \quad \text{in } R$$

$$u_i = f \qquad \text{on } S \quad (i=0, 1, 2, \cdots\cdots)$$

or equivalently

$$u_{i+1}(s) = h(s) - b\int G(s, t) u_i^n(t) u_{i+1}^n(t) dt. \tag{4}$$

For this sequence one may assert that the u_i are positive in R except perhaps for a set of measure zero on which $u_i^n=0$ (*i. e.*, $u_i=0$) [1].

To demonstrate the convergence of the iteration, it is shown below that

$$\|v_{i+1}\| \leqq r\|v_i\| \ (0<r<1), \text{ where } v_i=u_{i+1}^n-u_i^n.$$

Thus

$$\|v_{i+1}\| \leqq r^{i+1}\|v_0\|.$$

Here the norm is that of real L_2-space:

$$\|v\|=[\int v^2(t)dt]^{\frac{1}{2}}.$$

$\lim_{i\to\infty} u_i^n=u$ exists as the sum of the convergent in the series

$$u_0+\sum_{i=0}^{\infty} v_i.$$

From (4),

$$u_{i+2}-u_{i+1}=-b\int G(s, t)[u_{i+1}^n u_{i+2}^n(t)-u_i^n u_{i+1}^n(t)]dt$$

$$=-b\int G(s,t)u_{i+1}^n(v_{i+1}+v_i)(t)dt.$$

Therefore

$$v_{i+1}=-b[u_{i+2}^{n-1}+\cdots\cdots+u_{i+1}^{n-1}]\int G(s, t)u^n_{i+1}(v_{i+1}+v_i)(t)dt \quad (5).$$

Let $\{\varphi_j\}$ be the set of normalized characteristic solutions and the numbers λ_j, the corresponding characteristic values of

$$\int G(s, t)u_{i+1}^n(t)\varphi_j(t)dt=\lambda_j\varphi_j(s).$$

The existence and completeness of the φ_j and λ_j under certain continuity conditions on u_{i+1}^n (*i. e.*, u_{i+1}) is demonstrated in [2].

The corresponding differential equation for φ_j is

$$\lambda_j\Delta\varphi_j=-u_{i+1}^n\varphi_j \text{ in } R$$

$$\varphi_j=0 \qquad \text{on } S.$$

Here we have

$$\lambda_j\int\varphi_j\Delta\varphi_j dt=-\int u_{i+1}^n\varphi_j^2 dt.$$

By Green's theorem

$$-\lambda_j\int(\nabla\varphi_j)^2 dt=-\int u_{i+1}^n\varphi_j^2 dt,$$

so that, $\lambda_j \geqq 0$.

A bound on the λ_j is obtained from the integral form:

$$\lambda_j |\varphi_j(s)| = |\int G(s,\ t) u_{i+1}^n(t) \varphi_j(t) dt|$$

$$\leqq max|\varphi_j| \cdot max|u_{i+1}^n| \cdot max|\int G(s,\ t)dt|$$

$$= G_1 max|\varphi_j|.$$

It follows that $\lambda_j \leqq G_1$.

By the completeness of the φ_j, there exist constants $a_{i,j}$ and $a_{i+1,j}$ for which

$$v_i = \sum_{j=1}^{\infty} a_{i,\ j}\varphi_j, \quad \text{and} \quad v_{i+1} = \sum_{j=1}^{\infty} a_{i+1,\ j}\varphi_j.$$

Inserting the series forms for v_i and v_{i+1} into (5) gives

$$\sum_{j=1}^{\infty}\{a_{i+1,\ j}[1+b\,\lambda_j(u_{i+2}^{n-1}+\cdots+u_{i+1}^{n-1})]+a_{i,\ j}b\lambda_j[u_{i+2}^{n-1}+\cdots+u_{i+1}^{n-1}]\}\varphi_j=0.$$

From the orthogornality of the φ_j,

$$a_{i+1,\ j} = \frac{-a_{i,\ j}b\,\lambda_j[u_{i+2}^{n-1}+\cdots+u_{i+1}^{n-1}]}{1+b\,\lambda_j[u_{i+2}^{n-1}+\cdots+u_{i+1}^{n-1}]}.$$

Since $0 \leqq \lambda_j \leqq G_1$ and $u_{i+2}^{n-1}+\cdots+u_{i+1}^{n-1} \leqq n$,

$$0 \leqq \frac{b\lambda_j[u_{i+2}^{n-1}+\cdots+u_{i+1}^{n-1}]}{1+b\lambda_j[u_{i+2}^{n-1}+\cdots+u_{i+1}^{n-1}]} \leqq \frac{nbG_1}{1+nbG_1} = r < 1.$$

So that $$|a_{i+1,\ j}| \leqq r|a_{i,\ j}|.$$

By Parseval's theorem, we have

$$\|v_{i+1}\| \leqq r\|v_i\|.$$

Here r is independent of i. Hence the sequence u_i^n converges in the mean. Therefore u_i converges in the mean, as was to be shown.

References

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