A Note on de Bruijn's Number System

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1. INTRODUCTION. By the expression

$$N=A_1+A_2+\cdots\cdots=\sum_{i=1}^{\infty} A_i,$$

where $N=\{0, 1, 2,\cdots\cdots\}$, we shall mean that for the subsets A_i of N each natural number n can be expressed *uniquely* as a sum $n=a_1+a_2+\cdots\cdots$, where $a_i\in A_i$. Note that $A_i\cap A_j=\{0\}$ for $i\neq j$. Now let $1=p_1, p_2, p_3,\cdots\cdots$ be either an infinite or a finite sequence of positive integers such that $p_{i+1}=m_i p_i$ with integer $m_i>1$, and let $[a_i, b_i]\ p_i$ or $[a_i, \infty]\ p_i$ denote the set for $a_i<b_i$

$\{a_i p_i, (a_i+1)p_i, (a_i+2)p_i,\cdots,b_i p_i\}$ or $\{a_i p_i, (a_i+1)p_i,\cdots\cdots\}$, respectively.

de Bruijn(see [1]), according to our notations, has shown in fact

(1)
$$N=\sum_{i=1}^{\infty} [0, (m_i-1)]\ p_i \qquad \text{or}$$
$$N=\sum_{i=1}^{k} [0, (m_i-1)]\ p_i, \qquad \text{where } m_k=\infty.$$

The purpose of this note is to extend the above decomposition of N over Z, the set of all integers.

2. THEOREM 1. *If we choose a_i and with $b_i-a_i=m_i-1$ only to satisfy*

$$\sum_{i=1}^{\infty} a_i p_i=-\infty \quad \textit{and} \quad \sum_{i=1}^{\infty} b_i p_i=+\infty$$

(e.g. $a_{2i-1}=0$, $b_{2i}=0$ *for all* $i\geqq 1$), *then*

(2)
$$Z=\sum_{i=1}^{\infty} [a_i, b_i]\ p_i.$$

Proof. We claim that the set, denoted by $Z(t)$, of integers z in the range

$$\sum_{i=1}^{t} a_i p_i\leqq z\leqq \sum_{i=1}^{t} b_i p_i$$

can be expressed as

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$$Z(t)=\sum_{i=1}^{t}\,[a_i,\ b_i]\ p_i. \tag{3}$$

If so (2) follows from this by

$$Z=\lim_{i\to\infty} Z(t)=\sum_{i=1}^{\infty}\,[a_i,\ b_i]\ p_i.$$

Since by the definition of $Z(t)$ the right hand side is clearly a subset of the left hand side of (3), it suffices to show that the cardinalities of the both are the same. First, the cardinality of $Z(t)$ is

$$\begin{aligned}&1+\sum_{i=1}^{t} p_i(b_i-a_i)\\&=1+(m_1-1)+m_1(m_2-1)+m_1m_2(m_3-1)+\cdots\cdots+m_1\cdots m_{t-1}(m_t-1)\\&=m_1m_2\cdots\cdots m_t=p_{t+1}\end{aligned}$$

and the cardinality of the right hand side is also $m_1m_2\cdots\cdots m_t=p_{t+1}$, provided that all the numbers expressed there are distinct. Hence it remains to show that if

$$\sum_{i=1}^{t} p_ic_i=0 \text{ with } |c_i|<m_i$$

then $c_i=0$ $(1\leqq i\leqq t)$. But this is clear, for if $c_q\neq 0$ and $c_i=0$ $(q<i\leqq t)$ then we have

$$\left|\sum_{i=1}^{q-1} p_ic_i\right|=|p_qc_q|$$

but at the same time it is easy to see

$$\left|\sum_{i=1}^{q-1} p_ic_i\right|<p_q\leqq|p_qc_q|.$$

REMARKS 1. While (1) is the only form of decomposition for N, (2) is not necessarily so for Z.

2. $Z=A_1+A_2+\cdots\cdots$ implies $Z=-A_1-A_2-\cdots\cdots$ and $Z=(z_1+A_1)+(z_2+A_2)+\cdots\cdots$, where z's are arbitrary integers and only a finite number of them are nonzero.

3. By putting $n=p_{t+1}$ in (3) we obtain a unique sum decomposition of Z_n, the set of integers modulo n.

3. Furthermore, it might be of some interest to mention that for the sets

$$N^n=\{1^n, 2^n, 3^n, \cdots\cdots\} \quad \text{for } n>1$$

or

$$Z^n=\{z^n\,|\,z\in Z\} \quad \text{for an odd } n>1$$

there does not exist a similar unique sum decomposition we are looking for. These are two special cases of the following negative

THEOREM 2. $\quad S=\{Q(z)=\sum_{j=0}^{n} a_jz^j \mid a_n\neq 0 \text{ and } z\in Z'\}\ (n>1),$

where Z' is any infinite subset of Z, has no nontrivial unique sum decompositions. (A trivial one is $S=S+\{0\}+\{0\}+\cdots\cdots$.)

Proof. Assume there exists a nontrivial decomposition. Then we would have the following relation:

$$Q(z_0)+Q(z_i)=Q(z_i')$$

which is satisfied by infinitely many pairs (z_i, z_i') for a fixed z_0. This can be expressed as

$$(z_i'-z_i)\frac{Q(z_i')-Q(z_i)}{z_i'-z_i}=Q(z_0).$$

Here either factor of the left hand side must be an integral divisor of the fixed integer $Q(z_0)$ and in the second factor on the left hand side, the dominating term in size as either one of $|z_i|$ and $|z_i'|$ becomes large is

$$a_n\frac{z_i'^n-z_i^n}{z_i'-z_i}=a_n(z_i'^{n-1}+z_i'^{n-2}z_i+\cdots\cdots+z^{n-1})$$

But since when z_i' and z_i have the same sign the number of divisors with the form of the second factor is clearly limited, and also when they have different signs the number of divisors of $Q(z_0)$ with the form of the first factor is finite, there cannot exist infinitely many pairs (z_i, z_i') which satisfy the relation.

REMARKS 1. If $n=1$ and $Z'=Z$ in the theorem, then we have a desired decomposition

$$S=(a_1A_1+a_0)+a_1A_2+a_1A_3+\cdots\cdots,$$

derived from a decomposition of $Z=A_1+A_2+\cdots\cdots$.

2. If $Z=\sum A_i$ then obviously $\sum A_i'$, where each A_i' is a subset of A_i, is a decomposition of the set it expresses. For example, in (1) if we let $m_i=10$ for all i and let d be any digit with $1\leqq d\leqq 9$, then $\sum_{i=0}^{\infty}\{0, 10^i d\}$, where $\{0, 10^i d\}$ is a subset of $[0, (m_i-1)]p_i$, forms the set of all natural numbers expressed only by the digits 0 and d in the decimal representation. Thus the set of all natural numbers expressed by the digits 0 and d alone has a decomposition.

3. An infinite set of prime numbers has no nontrivial decomposition, unless there exists an infinitude of twin primes.

References

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