On Sub-H-Groups of an H Group and their Duals

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1. Introduction. An H space consists of a pointed topological space P together with a continuous multiplication $\mu: P\times P\longrightarrow P$ for which the constant map $c: P\longrightarrow P$ is a homotopy identity, i.e., $\mu(1_P, c)\simeq 1_P$, and $\mu(c, 1_P)\simeq 1_P$. An H group is an H space whose multiplication is homotopy associative and has a homotopy inverse. An H cogroup consists of a pointed topological space Q together with a continuous comultiplication $\nu: Q\longrightarrow Q\vee Q$ having the homotopy identity and a homotopy inverse such that it is homotopy associative [2].

The object of this paper is to define a sub-H-group of an H group and its dual conception, a quotient-H-cogroup, and to give their properties.

2. Preliminaries. Given maps $f: X\longrightarrow Y$ and $g: X\longrightarrow Z$, we let $(f, g): X\longrightarrow Y\times Z$ be the map defined by $(f, g)(x)=(f(x), g(x))$; and given $f: X\longrightarrow Z$ and $g: Y\longrightarrow Z$, we let $(f, g): X\vee X\longrightarrow Z$ be the map defined by $(f, g)|X=f$ and $(f, g)|Y=g$.

All spaces are considered to be pointed topological spaces and all maps are considered to be base point preserving continuous functions.

In the homotopy category of pointed topological spaces and continuous maps, we adopt the definitions of a monomorphism and an epimorphism from [1] as follows:

A map $f: X\longrightarrow Y$ is a monomorphism iff the only pairs $g_1: Z\longrightarrow X$, $g_2: Z\longrightarrow X$ such that $fg_1\simeq fg_2$ are the homotopic ones: $g_1\simeq g_2$. A map $f': X\longrightarrow Y$ is an epimorphism iff the only pairs $h_1: Y\longrightarrow Z'$, $h_2: Y\longrightarrow Z'$ such that $h_1f'\simeq h_2f'$ are the homotopic ones: $h_1\simeq h_2$.

For any categroy $\mathscr{C}$, and an object Y of $\mathscr{C}$, we let π_Y (or π^Y) be a covariant (or contravariant) functor from $\mathscr{C}$ to the category of sets and functions which assigns to an object Z (or X) of $\mathscr{C}$ the set $\pi_Y(Z)=\hom_{\mathscr{C}}(Y, Z)$ or $\pi^Y(X)=\hom_{\mathscr{C}}(X, Y)]$ and to a morphism $h: Z\longrightarrow Z'$ [or $f: X\longrightarrow X'$] the function $h_\#: \hom_{\mathscr{C}}(Y, Z)\longrightarrow \hom_{\mathscr{C}}(Y, Z')$ [or $f_\#: \hom_{\mathscr{C}}(X', Y)\longrightarrow \hom_{\mathscr{C}}(X, Y)$] defined by $h_\#(g)=hg$ for $g: Y\longrightarrow Z$ [or $f^\#(g')=g'f$ *for* $g': X'\longrightarrow Y$].

The following results are known [2].

2-1. *If P (or Q) is an H group (or H cogroup), π^P (or π_Q) is a contravariant (or covariant) functor from the homotopy category of pointed topological spaces and continuous maps with walues in the category of groups and homomorphisms.*

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2-2. *Let α be a map of an H group P (or H cogroup Q) into an H group P' (or H cogroup Q'). Then $\alpha_{\#}$(or $\alpha^{\#}$) is a natural transformation from π^P (or $\pi_{Q'}$) to $\pi^{P'}$ (or π_Q) in the category of groups if and only if α is a homomorphism.*

2-3. *If P (or Q) is a pointed space such that π^P (or π_Q) takes values in the category of groups, then P (or Q) is an H group (or H cogroup). Furthermore, the group structure on $\pi^P(X)$ [or $\pi_Q(X)$] is the same as given in 2-1.*

2-4. *If P s an H space and Q is any H cogroup, then [Q;P] is an abelian group and the group structure is defined by the multiplication map in P.*

General notations and definitions can be found in [2].

3. Sub-*H*-groups of an *H* group. We will assume throughout this section that P is an H group with a base point p_0 having the multiplication $\mu: P\times P \longrightarrow P$ for which c and φ are the homotopy identity and the homotopy inverse, respectively.

3-1 Definition. A pointed subspace P' of an H group P with the same base point p_0 is called a sub-*H*-group of P if P' is itself an H group such that the inclusion map is a homomorphism.

It is immediate from the definition that given H group P itself and the one point space $\{p_0\}$ are sub-*H*-groups of P.

Given a topological group G, a subgroup C' of G is a sub-*H*-group of G when G is regarded as an H group.

Given a pointed space (Y, y_0), let (Y', y_0) be a pointed subspace. Then the loop space $\Omega Y'$ based at y_0 is a sub-*H*-group of the loop space ΩY based at y_0.

3-2 Theorem. *Suppose that a subspace P' of an H group P with the same base point p_0 is itself an H group with the same multiplication μ. Then P' is a sub-H-group of P.*

Proof. Since the equality implies the homtopicity, it is obvious that the inclusion map is a homomorphism.

The following lemma is immediate from the Theorem 2-2.

3-3 Lemma. *Let P' be a subspace of Pwith the same base point p_0 such that P' is itself an H group. P' is a sub-H-group of P if and only if, for the inclusion map $i: P' \subset P$, $i_{\#}$ a natural transformaton from $\pi^{P'}$ to π^P in the category of groups.*

3-4 Proposition. *Let P' be a sub-H-group of an H group P such that the incluslon map $i: P' \subset P$ is a monomrphism. For any pointed space X, $\pi^{P'}(X)$ can be imbedded as a su bgroup into the group $\pi^P(X)$.*

Proof. Since P' is a sub-*H*-group of P, from Lemma 3-3, $i_{\#}(X)$ is a homomorphism from the group $\pi^{P'}(X)$ into the group $\pi^P(X)$ for all pointed space X. And since i is a monomorphism, for any two maps f and g of X into P', *if* $\simeq ig$ implies $f\simeq g$; which shows that $i_{\#}(X)$ is a monomorphism.

3–5 PROPOSITION. *Let P be a pointed space and P' a subspace with the same base point as P. Suppose that, for any pointed space X, $\pi^P(X)$ is a group and $\pi^{P'}(X)$ can be imbedded as a subgroup inte $\pi^P(X)$ by $i^*(X)$. Then P' is a sub-H-group of an H group P.*

Proof. Since $\pi^P(X)$ and $\pi^{P'}(X)$ are groups for all pointed space X, by Theorem 2–3, P and P' are H groups. By the definition of i_* and $i_*(X)$ being a homomorphism for any pointed space X, we obtain that i_* is a natural transformation from $\pi^{P'}$ to π^P. Hence, by Lemma 3–3, P' is a sub-H-group of an H group P.

Combining 3–4 and 3–5, we obtain the following.

3–6 THEOREM. *Let P' be a pointed subspace of an H group P such that the inclusion map $i : P' \subset P$ is a monomorphism. P' is a sub-H-group of P if and only if $\pi^{P'}(X)$ can be imbedded as a subgroup into $\pi^P(X)$ by $i^*(X)$ for all pointed space X.*

3–7 COROLLARY. *A weak retract of P' of an H group P with the same base point is a sub-H-group of P if and only if $\pi^{P'}(X)$ can be imbecdded as a subgroup into $\pi^P(X)$ for all pointed space X.*

Now, we will prove another necessary and sufficient conditions for a weak retract of an H group to be a sub-H-group. We first prove the following.

3–8 PROPOSITION. *If P' is a sub-H-group of an H group P, then*

(*a*) *there exists a continuous multiplication $\mu' : P' \times P' \longrightarrow P'$ sueh thct $i\mu' \simeq \mu(i \times i)$*

(*b*) *for the constant map $c' : P' \longrightarrow P'$, $ic' = ci$, and*

(*c*) *there exists a continuous map $\varphi' : P' \longrightarrow P'$ suht hat $i\varphi' \simeq \varphi i$.*

Proef. From the Definition 3–1, (*a*) and (*b*) are immediately obtained. It remains to show (*c*). Let X be an arbitrary pointed space, $[f] \in \pi^{P'}(X)$ and φ' be the homotopy inverse for μ'. By Lemma 3–3, $i_*(X)$ is a homomorpism from the group $\pi^{P'}(X)$ into the group $\pi^P(X)$; hence we have

$$i_*(X)([f]^{-1}) = (i_*(X)[f])^{-1} = [if]^{-1} = [\varphi i f],$$

and

$$i_*(X)([f]^{-1}) = i_*(X)([\varphi' f]) = [i\varphi' f],$$

$$\text{i.e., } [\varphi i f] = [i\varphi' f].$$

Since this fact holds for all pointed space X, we can take X as P' and f as $1_{P'}$. Then $[\varphi 1_{P'}] = [\varphi i]$ and $[i\varphi' 1_{P'}] = [i\varphi']$; which shows that $i\varphi' \simeq \varphi i$.

For converse, we have to give a restriction for the inclusion to be a monomorphism. Thus we obtain the following.

3–9 PROPOSITION. *Let P' be a pointed subspace of an H group P. Suppose that the conditions (a), (b) and (c) given in 3–7 are satisfied and the inclusion map $i : P' \subset P$ is a monomorphism. Then P' is a sub-H-group of P.*

Proof. From (*a*) and (*b*), we have

$$\begin{aligned} & i\mu'(1_{P'}, c') \simeq \mu(i \times i)(1_{P'}, c') \\ & = \mu(i\, 1_{P'}, ic') = \mu(1_P i, ci) \\ & = \mu(1_P, c)i \simeq 1_P i = i\, 1_{P'}. \end{aligned}$$

Since i is a monomorphism, we have $\mu'(1_{P'}, c') \simeq 1_{P'}$. Replacing $(1_{P'}, c')$ with $(c', 1_{P'})$ we also have $\mu'(c', 1_{P'}) \simeq 1_{P'}$. Thus we have proved that c' is a homotopy identity for μ'. Next, using (a) again, we have

$$\begin{aligned} & i\mu'(\mu' \times 1_{P'}) \simeq \mu(i \times i)(\mu' \times 1_{P'}) = \mu[(i\mu') \times (i\, 1_{P'})] \\ & \simeq \mu[((\mu(i \times i)) \times 1_P i] = \mu(\mu \times 1_P)(i \times i \times i) \\ & \simeq \mu(1_P \times \mu)(i \times i \times i) = \mu[(1_P i) \times ((\mu(i \times i))] \\ & \simeq \mu[(i\, 1_{P'}) \times (i\mu')] = \mu(i \times i)(1_{P'} \times \mu') \simeq i\mu'(1_{P'} \times \mu'); \end{aligned}$$

which shows that μ' is homotopy associative. Now, from (c) we have

$$\begin{aligned} & ic' = ci \simeq \mu(1_P, \varphi)i = \mu(1_P i, \varphi i) \simeq \mu(i\, 1_{P'}, i\varphi') \\ & = \mu(i \times i)(1_{P'}, \varphi') \simeq i\mu'(1_{P'}, \varphi'); \end{aligned}$$

which shows that $c' \simeq \mu'(1_{P'}, \varphi')$. Replacing $(1_P, \varphi)$ with $(\varphi, 1_P)$, we also have $c' \simeq \mu'(\varphi', 1_{P'})$. Thus we have proved that φ' is a homotopy inverse for μ'. Therefore P' is an H group. However, from (b), P' has the same base point as P; and from (a) again, i is a homomorphism. Hence, P' is a sub-H-group of P.

Combining 3-7 and 3-8, we have the following.

3-10 Theorem. *Let P' be a pointed subspace of an H group P such that the inclusion map $i : P' \subset P$ is a monomorphism. P' is a sub-H-group of P if and only if the conditions (a), (b) and (c) given in* 3-7 *are satisfied.*

3-11 Corollary. *A weak retract of P of an H group P is a sub-H-group of P if and only if the conditions (a), (b) and (c) given in* 3-7 *are satisfied.*

The following sufficient condition also holds:

3-12 Proposition. *Let P' be a subspace of P with the same base point such that P is deformable into P'. Then P' is a sub-H-group of P.*

Proof. Since P is deformable into P', the inclusion map $i : P' \subset P$ has a right homotopy inverse $r : P \longrightarrow P'$. Let μ', c' and φ' be maps defined by

$$\mu' = r\mu(i \times i), \quad c' = rci \quad \text{and} \quad \varphi' = r\varphi i.$$

Then they are all well defined continuous maps and they satisfy the conditions in 3-7. For,

(a) $i\mu' = ir\mu(i \times i) \simeq 1_P \mu(i \times i) = \mu(i \times i)$,

(b) $ic' = irci \simeq 1_P ci = ci$,

(c) $i\varphi' = ir\varphi i \simeq 1_P \varphi i = \varphi i$.

However, since P' has the same base point as P, in (b), we have $ic' = ci$.

4. Quotient-H-cogroups of an H cogroup. We will assume throughout this section that Q is an H cogroup with a base point q_0 having the comultiplication $\nu : Q \longrightarrow Q \vee Q$ for which c and ϕ are the homotopy identity and the homotopy inverse, respectively.

Let R be an equivalence relation in Q, $Q' = Q/R$ the quotient space obtained by R, and $\xi : Q \longrightarrow Q'$ the canonical map.

4-1 DEFINITION. *A* quotient space $Q' = Q/R$ of an H cogroup Q is called a quotient-H-cogroup of Q if Q' is itself an H cogroup such that ξ is a homomorphism.

The following Propositions 4-2 and 4-3, Theorem 4-4 are immediately obtained from the Propositions 3-4, 3-5 and Theorem 3-6, by duality.

4-2 PROPOSITION. *Let Q' be a quotient-H-cogroup of an H cogroup Q such that the canonical map $\xi : Q \longrightarrow Q'$ is an epimorphism. For any pointed space X, $\pi_Q{}'(X)$ can be imbedded as a subgroup into the group $\pi_Q(X)$ by $\xi^{\#}(X)$.*

4-3 PROPOSITION. *Let Q be a pointed space, Q' a quotient space of Q with respect to equivalence relation and ξ the canonical map of Q onto Q'. Suppose that for any pointed space Y, $\pi_Q(X)$ is a group and $\pi_{Q'}(X)$ can be imbedded as a subgroup into $\pi_Q(X)$ by $\xi^{\#}(X)$. Then Q' is a quotient-H-cogroup of an H cogroup Q.*

4-4 THEOREM. *Let Q' be a quotient space of an H cogroup Q with respect to an equivalence relation such that the canonical map $\xi : Q \longrightarrow Q'$ is an epim orphism. Q' is a quotient-H-cogroup of Q if and only if $\pi_{Q'}(X)$ can be imbedded as a subgroup into $\pi_Q(X)$ by $\xi^{\#}(X)$ for all pointed space X.*

The following Propositions 4-5, 4-6 and Theorem 4-7 are immediately obtained from the Propositions 3-7, 3-8 and Theorem 3-9 by duality.

4-5 PROPOSITION. *If Q' is a quotient-H-cogroup of an H-cogroup Q, then*

(*a*) *there exists a continuous comultiplication $\nu' : Q' \longrightarrow Q' \vee Q'$ such that $\nu'\xi \simeq (\xi \vee \xi)\nu$.*

(*b*) *for the constant map $c' : Q' \longrightarrow Q'$, $c'\xi = \xi c$, and*

(*c*) *there exists a continuous map $\phi' : Q' \longrightarrow Q'$ such that $\phi'\xi \simeq \xi\phi$.*

4-6 PROPOSITION. *Let Q' be a quotient space of an H cogroup Q with respect to an equivalence relation such that the canonical map $\xi : Q \longrightarrow Q'$ is an epimorphism. Suppose that the conditions* (*a*), (*b*) *and* (*c*) *given in* 4-5 *are satisfied. Then Q' is a quotient-H -cogroup of Q.*

4-7 THEOREM. *Let Q' be a quotient space of an H cogroup Q with respect to an equivalence relation such that the canonical map $\xi : Q \longrightarrow Q'$ is an epimorphism. Q' is a quotient-H-cogroup of Q if and only if the conditions* (*a*), (*b*) *and* (*c*) *given in* 4-5 *are satisfied.*

Finally, we will prove a theorem relating to a sub-H-group and a quotient-H-cogroup.

4-8 THEOREM. Let *Q' be a quotient-H-cogroup of an H cogroup Q such that the canonical map $\xi : Q \longrightarrow Q'$ is an epimorphism and P' is a sub-H-group of an H group P*

such that the inclusion map $i: P' \subset P$ *is a monomorphism. Then* $[Q'; P']$ *can be imbedded as an abelian subgroup into* $[Q: P]$.

Proof. From the Theorem 2-4, $[Q'; P']$ is an abelian group and its group structure is defined by the multiplication map in P'. Hence, if $*$ and $*'$ are group operations in $[Q; P]$ and $[Q'; P']$, respectively, and if μ and μ' are the multiplication maps for P and P', respectively, then for the pairs $[f]$, $[g]$ belonging to $[Q; P]$ and $[f']$, $[g']$ belnoging to $[Q'; P']$, we have

$$[f]*[g] = [\mu(f, g)], \quad [f]*'[g'] = [\mu'(f', g')].$$

Define a map $\lambda: [Q'; P'] \longrightarrow [Q; P]$ by $\lambda[f] = [if\xi]$ for $[f] \in [Q'; P']$. Since $if\xi$ is continuous and $if\xi \simeq ig\xi$ is implied from $f \simeq g$, λ is well defined map. And we have

$$\begin{aligned} &\lambda([f] *' [g]) = \lambda([\mu'(f, g)]) = [i\mu'(f, g)\xi] \\ &= [\mu(i \times i)(f, g)\xi] = [\mu(if\xi, ig\xi)] \\ &= [if\xi] * [ig\xi] = (\lambda[f]) * (\lambda[g]); \end{aligned}$$

which shows that λ is a homomorphism. Since i is a monomorphism and ξ an epimorphism, $\lambda[f] = \lambda[g]$, i.e., $if\xi \simeq ig\xi$, implies that $f \simeq g$, i.e., $[f] = [g]$; which shows that λ is a monomorphism.

References

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