Stone-Čech Compactifications of Infinite Discrete Spaces

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1. Introducton. This paper is a study of the nature of the Stone-Čech compactification βX of an infinite discrete space X. If X is an infinite discrete space, then it is clear that X contains as many disjoint topological copies of itself as there are points in X. From this, we deduce that βX also has this property, that is, βX contains as many disjoint copies of itself as there are points in βX. Some sufficient conditions that subsets of βX be countably compact are also given.

2. Preliminary concepts. Let N and R denote the space of natural numbers and the space of real numbers, respectively. The cardinality of a set X is denoted by $|X|$, with the exception that we use $\aleph_0$ and c in place of $|N|$ and $|R|$, respectively.

Let X be a space. The closure in X of a subset S of X is denoted by $cl_X S$. If X is a completely regular T_1-space we denote by $C^*(X)$ the set of bounded continuous functions from X into R. A subset S of X is C^*-embedded in X if every function in $C^*(S)$ extends to a function in $C^*(X)$. Thus X is C^*-embedded in the Stone-Čech compactification βX of X. It will be convenient to recall here that βX is characterized by the property of being a compact Hausdorff space having X as a C^*-embedded dense subset. From this, it follows immediately that a subset S of X is C^*-embedded in X if and only if βS is identical with the closure in βX of S. We also note that the Tietze extension theorem is phrased as: Closed sets are C^*-embedded in a normal space.

3. Disjoint copies of βX in $\beta X - X$. As mentioned in the introduction, the purpose of this section is to prove that the Stone-Čech compactification of an infinite discrete space contains as many disjoint copies of itself as its cardinality. To do this, the following lemma will be needed.

LEMMA 1. *If a regular space is the sum of a discrete open subset and a countable set then it is paracompact.*

Proof. Let X be a regular space which is the sum of a discrete open subset Y and a countable set Z, and let $\mathcal{U}$ be any open cover of X. For each point z of Z, pick a member U_z of $\mathcal{U}$ with $z \in U_z$. Then the collection consisting of these U_z and the points not lying on any of the sets U_z is an open cover of X as Y is a

Received by the editors December 17, 1968.

discrete open subset of X. Since this open cover is readily seen to be a σ-discrete refinement of $\mathcal{U}$, X must be paracompact by [3, Theorem 5.28].

REMARK 1. In Lemma 1, one cannot discard the requirements that Y be open or that Z be countable. To see this, let D be a discrete subset of $\beta N-N$ with cardinality c. Such D exists since $\beta N-N$ contains c many disjoint open sets. If D is C^*-embedded in $N\cup D$ then it must be C^*-embedded in βN as well. However, this will imply that the cardinality of βN is not less than that of βD, which is impossible by Lemma 3 below. Accordingly, $N\cup D$ is not paracompact. Regarding D to be Y, one concludes that Lemma 1 fails if Y is not required to be open. By interchanging Y and Z, we also see that one cannot dispense with the condition that Z be countatle.

THEOREM 1. *If X is an infinite discrete space then there is a family $\mathfrak{F}$ consisting of pairwise disjoint subsets of $\beta X-X$ such that each member of $\mathfrak{F}$ is homeomorphic with βX and $|\mathfrak{F}|=|\beta X|$.*

Proof. Decompose X into disjoint subsets X_α, $\alpha\in A$, where the X_α's and A are equipotent with X. Each X_α is C^*-embedded because it is closed in the normal space X. Hence we have $cl_{\beta X}X_\alpha=\beta X_\alpha$. Moreover, the sets $cl_{\beta X}X_\alpha$ are open and closed in βX as βX is the disjoint sum of βX_α and $\beta(X-X_\alpha)$, each of which is a homeomorph of βX. We also remark that the sets $cl_{\beta X}X_\alpha$ are all disjoint since any two of them are completely separated in X.

For each α in A let h_α be a homeomorphism of βX into $cl_{\beta X}X\alpha$, and let Y_p be the set of points $h_\alpha(p)$, $\alpha\in A$, for $p\in\beta X-X$. If p and q are distinct points of $\beta X-X$ then $X\cup Y_p\cup Y_q$ is the disjoint sum of the sets X_α added with the points $h_\alpha(p)$ and $h_\alpha(q)$. Since each $cl_{\beta X}X_\alpha$ is open and closed in βX each of these summands must be open and closed in $X\cup Y_p\cup Y_q$. Hence $X\cup Y_p\cup Y_q$ is paracompact by [1, 1-9] as the sets $X_\alpha\cup h_\alpha(p)\cup h_\alpha(q)$ are paracompact by Lemma 1. On the other hand, since the sets Y_p and Y_q share only one point in common with each $X_\alpha\cup h_\alpha(p)\cup h_\alpha(q)$, they are discrete and closed in $X\cup Y_p\cup Y_q$. Also it is clear that Y_p and Y_q are disjoint because h_α is a homeomorphism. Thus, it follows that $cl_{\beta X}Y_p$ and $cl_{\beta X}Y_q$ are disjoint as they are disjoint closed subsets of the normal space $X\cup Y_p\cup Y_q$ and $X\cup Y_p\cup Y_q$ must be C^*-embedded in βX.

Now, it is clear from the above arguments that the $cl_{\beta X}Y_p$ are disjoint subsets of βX homeomorphic with βX. They are all contained in $\beta X-X$ as each Y_p is disjoint from the open subset X of βX. It is enough for our purpose to let that $\mathfrak{F}$ be the collection of sets $cl_{\beta X}Y_p$, $p\in\beta X-X$. That $|\mathfrak{F}|=|\beta X|$ follows from Lemma 3.

4. Countably compact subsets of βX. In this section, we investigate conditions under which a subset of βX be countably compact. The fact that infinite closed subsets of βX contain homeomorphic images of βN is essential in our argument.

We begin by reviewing some known results.

LEMMA 2. *If X is an infinite discrete space then countable sets of βX are C^*-embedded in βX.*

This lemma can be proved by using Lemma 1. In fact, if Z is a countable subset of βX then $X \cup Z$ is normal by Lemma 1 as $X - Z$ is a discrete open subset of $X \cup Z$, and it must be C^*-embedded in βX as well because $X \cup Z$ is C^*-embedded in βX.

The reader is refered to [2] for the proof of the following result which is essentially due to Hausdorff.

LEMMA 3. *If X is an infinite discrete space, then $\beta X = 2^{2^{|X|}}$.*

Now let F be any infinite closed subset of βX. Since F contains a countably infinite discrete subset, F contains a copy of βN by Lemma 2. This together with Lemma 3 will imply

LEMMA 4. (*Gillman-Jerison-Henriksen*). *If X is an infinite discrete space then every infinite closed subset of βX contains a homeomorphic image of βN, and hence its cardinality is greater than that of the continuum.*

We are now ready to state

THEOREM 2. *Let X be an infinite discrete space. If Y is a subset of βX whose cardinality is not greater than that of the continuum, then $\beta X - Y$ is countably compact.*

Proof. Let Z be a countably infinite subset of $\beta X - Y$. Then the closure in $\beta X - Y$ of the set Z is $cl_{\beta X} Z - Y$, and its cardinality is greater than c by Lemma 4. This means that $cl_{\beta X - Y} Z$ is not empty. This proves the theorem.

REMARK 2. One might be tempted to assert that βX remains to be countably compact with $|X|$ many points deleted. Unfortunately, this is not the case as we are now going to clarify.

Let X be a discrete space with cardinality at least 2^c, and let Y be a countably infinite subset of X. Then βX is the topological sum of $\beta(X - Y)$ and. βY. The space $\beta(X - Y) \cup Y$ is the complement in βX of the set $\beta Y - Y$ whose cardinality does not exceed $|X|$. In fact, $\beta Y - Y$ has the cardinality 2^c by Lemma 3. However, $\beta(X - Y) \cup Y$ is not countably compact becase it has the infinite discrete closed set Y.

REMARK 3. It is known that there is a subset G of $\beta N - N$ such that every infinite subset of $\beta N - N$ has a limit point in G. It follows that every subset of βN containing G is countably compact. However, Theorem 2 is independent from this result as Y may well intersect G.

REMARK 4. Theorem 2 implies that $\beta N - N$ contains a countably compact subset which fails to be locally compact at each point. Note also that βX fails to contain a homeomorphic image of the space Q of rational numbers. This is true because $\beta Q - Q$ has an infinite discrete closed subset [2, 9C].

We now proceed to show that βX may remain to be countably compact by omission of certain types of subsets with cardinality greater than c. To do this, we need the following result which is apparently known.

LEMMA 5. *If X is an infinite discrete space, then there is a family of $\mathscr{G}$ of open and closed subsets of $\beta X - X$ such that each member of $\mathscr{G}$ is homeomorphic with $\beta X - X$ and $|\mathscr{G}| = |X|\aleph_0$.*

A proof of this lemma is outlined in [2] for $X = N$. If X is uncountable, we need only decompose X into $|X|$ many disjoint copies X_α of itself and let $\mathscr{G}$ be the family of the sets $\beta X_\alpha - X_\alpha$.

THEOREM 3. *Let X be an infinite discrete space. If Y is a subset of βX which is the countable sum of homeomorphic images of βN, then $\beta X - Y$ is countably compact.*

Proof. By hypothesis, Y can be expressed as the countable sum of sets βZ_n, where each Z_n is a homeomorphic image of N in βX. It suffices to show that if D is a countably infinite discrete subset of $\beta X - Y$ then D has a limit point not lying in Y. To do this, let Z_n' denote the intersection of Z_n with $\beta D = cl_{\beta X} D$. Then $\beta Z_n'$ is a subset of $\beta D - D$. We also observe that βZ_n is the disjoint sum of $\beta Z_n'$ and $\beta(Z_n - Z_n')$.

Since $\beta(Z_n - Z_n')$ must be disjoint from βD by Lemma 2, each βZ_n meets $\beta D - D$ in $\beta Z_n'$. It follows that $Y \cap (\beta D - D)$ is the countable sum $\cup \beta Z_n'$ which has the countable dense subset $\cup Z_n'$. On the other hand, since $\beta D - D$ fails to have a countable dense set by Lemma 5. It follow sthat $\beta D - D$ is not identical with $Y \cap (\beta D - D)$. In other words, $(\beta D - D) - Y \neq \phi$ i.e., D has a limit point as desired.

An immediate consequence of Theorem 3 is

COROLLARY. *If X is an infinite discrete space then $\beta X - X$ is not the countable sum of images of βN.*

Proof. If otherwise X would be countably compact.

References

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