Rings of Continuous Functions with Realcompact Supports

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1. Introduction. This paper is devoted to investigating properties of the set $C_R(X)$ of real functions continuous on a space X having realcompact supports. The main result is that if the supports are C-embedded, then $C_R(X)$ agrees with the intersection of all prime ideals which are contained in free real maximal ideals. It is also shown that in normal spaces, $C_R(X)$ is a free ideal if and only if X is locally realcompact, and the latter statement is equivalent to the requirement that X be open in its realcompactification.

It will be convenient to recall some definitions and notation that are relevant to the contents of this note. As usual, the ring of continuous real functions on a space X is denoted by $C(X)$. A maximal ideal M of $C(X)$ is said to be real if the factor ring of $C(X)$ modulo M is isomorphic to the real number field. The space X is realcompact if each real maximal ideal M is fixed, i.e., there is a point of X at which each function in M vanish. X is a P-space if every prime ideal of $C(X)$ is maximal. In this note, the word "space" is used to mean a Tychonoff space.

2. C-embedded realcompact sets. Except for the first Lemma, this section is independent from the other parts of the present paper. Theorem 1, which may be of some interest in itself, was obtained when we tried to generalize the main results to nonnormal spaces.

LEMMA 1. *Finite union of C-embedded realcompact subsets of a space X is realcompact.*

Proof. Let A and B be two realcompact C-embedded sets. Then $cl_{\upsilon X}A = \upsilon A = A$ and $cl_{\upsilon X}B = \upsilon B = B$, and we have

$$A \cup B = cl_{\upsilon X}A \cup cl_{\upsilon X}B = cl_{\upsilon X}(A \cup B).$$

It follows that $A \cup B$ is a closed subset of the realcompact space υX, hence is realcompact.

Union of two realcompact subsets need not be realcompact even if both are discrete and one is countable; consider the sets N and D in the space Ψ of Isbell described in [1, 5I].

If $\mathfrak{F}$ is a family of subsets of X and A is a subset of X, we denote by $\mathfrak{F} \cap A$ the collection of all sets $F \cap A$ where $F \in \mathfrak{F}$.

Received by the editors December 17, 1968.

LEMMA 2. *Let A be a C-embedded subset of X. In order that A be realcompact, it is necessary and sufficient that if $\mathfrak{F}$ is a z-ultrafilter on X such that $\mathfrak{F} \cap A$ has the countable intersection property, $\mathfrak{F} \cap A$ has nonempty intersection.*

Proof. Since A is C-embedded, every zero-set of A is of the form $Z \cap A$, where Z is a zero-set in X. So any real z-ultrafilter of A is of the form $\mathfrak{F} \cap A$ where $\mathfrak{F}$ is a z-ultrafilter in X and $\mathfrak{F} \cap A$ has the countable intersection property and vice versa. Hence the proof is complete by the definition of realcompactness.

A realcompact (not C-embedded) subset of X may not satisfy the condition of Lemma 2 even if it is a zero-set. In fact, if ψ is the space mentioned in the remark following Lemma 1, $\nu\psi$ coincides with $\beta\psi$ as ψ is pseudocompact. The subset D of ψ is realcompact zero-set and $cl_{\nu\psi}D$ is a compact set. This implies, however, that there is a free real maximal ideal M of $C(\nu\psi)$ such that $Z[M] \cap D$ has the countable intersection property as D is not compact.

We do not know whether there is a non-realcompact subset which satisfies the condition of Lemma 2.

THEOREM 1. *Let M be a real maximal ideal of $C(X)$. Then M is free if and only if for every C-embedded realcompact subset A of X, there is a function f in M such that $Z(f) \cap A = \phi$.*

Proof. Sufficiency: Since each point x is C-embedded and realcompact, there is a function f in M with $f(x) \neq 0$. That is, M is a free ideal.

Necessity: Assume there is a C-embedded realcompact, compact subset A of X such that for every f in M, $Z(f) \cap A \neq \emptyset$. Since M is a real maximal ideal, $Z[M]$ is closed under the formation of countable intersection, and by Lemma 2, there is a point in A at which each f in M vanishes. This contradiction proves the theorem.

3. Functions with realcompact supports. The support of a function f is defined by the closure of the cozero-set $Coz\,(f) = X - Z(f)$. Let $C_R(X)$ denote the subset of all functions in $C(X)$ with realcompact supports. By definition, $C_K(X) \subset C_R(X)$. Also it is clear that X is realcompact if and only if $C(X) = C_R(X)$.

THEOREM 2. *If X is normal, $C_R(X)$ is a z-ideal (not necessarily proper) of $C(X)$.*

Proof. Let f and g be functions in $C_R(X)$ and $C(X)$ respectively. Since $clCoz(fg) \subset clCoz(f) \cap clCoz(g)$, $clCoz(fg)$ is a closed subset of a realcompact subset $clCoz(f)$, hence is realcompact. Next, if f and g are functions in $C_R(X)$, then $clCoz\,(f-g)$ is contained in $clCoz(f) \cup clCoz(g)$ which is realcompact by Leamma 1. It follows that $clCoz(f-g)$ is realcompact. Finally, it is obvious that $Z^{\leftarrow}[Z[C_R(X)]] = C_R(X)$.

THEOREM 2′. *If X is a P-space, $C_R(X)$ is a z-ideal (not necessarily proper) of $C(X)$.*

Proof. In the previous theorem, normality is needed only to ensure that supports be C-embedded. Since every cozero-set is C-embedded and closed in a P-space by [1, 14.29], the proof of Theorem 2 will serve here almost word for word.

Theorem 2′ does not follow from Theorem 2 as there are nonnormal P-spaces. To see this, let W be the space of ordinals which are not greater than the first ordinal ω_2 with cardinality $\aleph_2$, let A be the subspace of W obtained by deleting all limit ordinals having countable bases, and let B be the subspace of A consisting of those ordinals which do not exceed the first uncountable ordinal ω_1. Then the space X obtained from $A \times B$ by deleting the point (ω_2, ω_1) is easily seen to be a P-space. However, it fails to be normal because there does not exist a function f in $C(X)$ which vanishes on $(A \times \omega_1) \cap X$ but equals 1 on $(\omega_2 \times B) \cap X$.

Recall that if P is a prime ideal, then there is a unique point q in βX with $O^q \subset P \subset M^q$. The zero-sets of P has the countable intersection property if and only *if* $q \in vX$, and in this case M^q is a real maximal ideal. Note also that P is free if and only if q is not in X. Hence $\bigcap_{p\in vX-X} O^p$ is the intersection of all free prime ideals with countable intersection property. By the Gelfand-Kolmogoroff Theorem, $f \in O^p$ if and only if $cl_{\beta X}Z(f)$ is a neighborhood of p in βX. It follows that $f \in O^p$ for some p in vX if and only if $cl_{vX}Z(f)$ is a neighborhood of p in vX. From these remarks, one sees that $f \in \bigcap_{p\in vX-X} O^p$ if and only if $cl_{vX}Z(f)$ is a neighborhood of $vX - X$. This leads us to the following.

THEOREM 3. *If X is normal, then $C_R(X)$ is identical with the intersection of all free prime idels of $C(X)$ whose zero-sets have the countable intersection property. That is, $C_R(X) = \bigcap_{p\in vX-X} O^p$.*

Proof. Let $clCoz(f)$ be realcompact. It is closed in vX as it is C-embedded in vX by the normality of X. It follows that $Z(f^v)(= cl_{vX}Z(f))$ is a neighborhood of $vX-X$, i. e., $f \in \bigcap_{p\in vX-X} O^p$.

Conversely, suppose that f lies in the intersection of all O^p, $p \in vX - X$. In this case, $Coz(f)$ coincides $Coz(f^v)$, and all we have to do is to prove that the support of f^v is a subset of X. This follows, however, from the fact that $Z(f^v)$ is a neighborhood of $vX - X$.

COROLLARY 1. *If X is a normal space, $C_R(X)$ is contained in the intersection of all free real maximal ideals.*

COROLLARY 2. *Let X be a normal space. A necessary and sufficient condition that the intersection of all free real maximal ideals be identical with the ring $C_R(X)$ is that the support of f be closed in vX whenever $Coz(f) = Coz(f^v)$.*

Proof. Necessity: If $Coz(f) = Coz(f^v)$, then f belongs to every M^p for $p \in vX-X$ and it lies in $C_R(X)$. Thus $clCoz(f)$ is realcompact andd C-embedded in vX. This implies that $clCoz(f)$ is closed in vX.

Sufficiency: Let f belong to every free real maximal ideal. By Gelfand-Kolmogoroff Theorem, $f^v(p) = 0$ for all p in $vX - X$ and we have $Coz(f) = Coz(f^v)$. Hence by hypothesis, $cl_X Coz(f)$ is realcompact. That is, $C_R(X)$ contains the intersection of all free real maximal ideals. This completes the proof.

THEOREM 4. *If X is a P-space, then $C_R(X)$ is identical with the intersection of all free real maximal ideals of $C(X)$, i.e., $C_R(X) = \bigcap_{p\in \upsilon X - X} M^p$.*

Proof. By the proofs of the Theorems 2′ and 3, we see that $C_R(X)$ is the intersection of the ideals O^p, $p\in \upsilon X - X$. Theorem 4 then follows immediately from the definition that prime ideals are maximal in a P-space.

4. Locally realcompact spaces. In this section, we characterize the normal spaces X for which $C_R(X)$ is a free ideal of $C(X)$. A space is said to be locally realcompact if each point has a realcompact neighborhood.

THEOREM 5. Let *X be a normal space, then the following are equivalent.*

(1) *X is locally realcompact.*

(2) *X is open in υX.*

(3) *$C_R(X)$ is a free ideal.*

Proof. (1) implies (2). Let U be an open neighborhood of a point x in X with realcompact closure. Then there is an open neighborhood V of x in υX such that $U = V\cap X$. Since $cl_X U$ is C-embedded in X, it is C-embedded in υX as well. Realcompactness of $cl_X U$ then implies that it is also closed in υX. On the other hand $cl_{\upsilon X} U = cl_{\upsilon X}(V\cap X) \supset V\cap cl_{\upsilon X} X = V$ since V is open in υX. Hence $cl_X U$ is a neighborhood of x in υX contained in X. Thus X is open in υX.

(2) implies (3). It suffices to show that $\bigcap_{p\in \upsilon X - X} O^p$ is free. For any point x of X, $\upsilon X - X$ is completely separated from $\{x\}$ since it is closed and υX is completely regular. Accordingly, there exists a function g in $C(\upsilon X)$ whose zero-set is a neighborhood of $\upsilon X - X$ not containing x. Now it follows that $g|X$ is in $\bigcap_{p\in \upsilon X - X} O^p$ and $(g|X)(x) = g(x) \neq 0$.

(3) implies (1). For any x in X, there is a function f in $C_R(X)$ such that $f(x) \neq 0$. $Coz(f)$ is an open neigborhood of x, and $cl_X Coz(f)$ is a realcompact neighborhood of x since f is in $C_R(X)$.

From conditions 1 and 2 of Theorem 5, we have

COROLLARY. *If X is locally real compact normal space, then $\upsilon X - X$ is realcompact.*

Reference

1. Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, Van Nostrand, 1960.

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