On the Pseudo-Riemannian Spaces

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Preliminary. We consider the usual Riemannian spaces whose metric is given as

(1) $$ds^2=g_{ij}dx^i dx^j, \quad i,j=1,\ 2, \cdots\cdots, \ n$$

and whose affine connection Γ^h_{ij} is asymmetric.

Now we put

(2) $$\Gamma^h_{ij}=\begin{Bmatrix} h \\ ij \end{Bmatrix}+\frac{1}{2}g^{hk}(T_{ijk}+T_{jik})+\frac{1}{2}T^h_{ij}$$

where $\{{}_i{}^h{}_j\}$ is the second symbol of Christoffel with respect to g_{ij} and the T^h_{ij} tensor represents the torsion of the spaces and non symmetric in i and j. The connection may be remained preservative as g_{ij}:

$$D_h g_{ij}=g_{ij;h}=0,$$

$$(D_h g_{ij}=\frac{\partial g_{ij}}{\partial x^h}-g_{ik}\Gamma^k_{jh}-g_{jk}\Gamma^k_{ih}).$$

Of course, $D_h g_{ij}$ is the absolute derivative of g_{ij} calculated in the space with the symmetric connection and $g_{ij;h}$ is the absolute derivative of g_{ij} in the Riemannian spaces without torsion.

Now we suppose the repère ([1], [2])

(3) $$\begin{cases}\overrightarrow{dM}=dx^i\vec{e}_i \\ d\vec{e}_i=\omega^k_i\vec{e}_k\end{cases}$$

where $\omega^k_i=\Gamma^k_{ih}dx^h.$

By choosing a map of the space to an Euclidean usual space, one can obtain a displacement attached on a simple cycle such that the last position of the repère(3) coincides with the initial. This displacement may be given by

(4) $$\begin{cases}\overrightarrow{\Delta M}=\Omega^i\vec{e}_i \\ \overrightarrow{\Delta e_i}=\Omega^k_i\vec{e}_k\end{cases}$$

where $\Delta=\delta d-d\delta.$

represents the difference of two symbols of differentiation d and δ. [1] And further applying successively to the vector elements of M and $\vec{e}_i$, we have the well known forms of the relations:

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$$(5)\qquad \Omega^i = T^i_{kh}[dx^k dx^h] = T^i_{kh}(\delta x^k dx^h - dx^k \delta x^h),$$
$$\Omega^k_i = R^k_{ihl}[dx^l dx^h]$$

where T^i_{kh} and R^k_{ihl} be respectively the tensors of torsion and curvature of the space:

$$(6)\qquad T^i_{kh} = \Gamma^i_{kh} - \Gamma^i_{hk},$$
$$R^k_{ihl} = \frac{\partial \Gamma^k_{il}}{\partial x^h} - \frac{\partial \Gamma^k_{ih}}{\partial x^l} + \Gamma^s_{il}\Gamma^k_{sh} - \Gamma^s_{ih}\Gamma^k_{sl}.$$

1. Pseudo-Riemannian mainfolds. A Riemannian manifold R is called a pseudo -Riemannian space R^* if the following properties a) and b) are satisfied:

a) The translation such that the origin of the final position of the repère (3) coincides with the origin of the initial in the displacement attached to a simple cycle in the tangent to the cycle.

b) The curvature tensor R_{ijhl} satisfies the fundamental conditions of the Riemannian space R:

$$(7)\qquad \begin{cases} R_{ijhl} = R_{hlij} \\ R_{ijhl} + R_{ihlj} + R_{iljh} = 0. \end{cases}$$

Under this definition of pseudo-Riemannian, many interesting results were obtained already (see [2], [3]). The author wants in this section to confine himself to point out some basic properties. The condition a) may be realized if we suppose

$$\overrightarrow{\Delta M} = [d\overrightarrow{M}\omega]$$

where ω is a Pfaffian form

$$\omega = V_k dx^k$$

and V_k the components of a vector. Hence we find

$$(8)\qquad T^i_{kh} = \delta^i_k V_h - \delta^i_h V_k$$

This is the particular case of the semi-symmetric torsion of Schouten.

At that time, one can find from (2) the components of the affine connection:

$$(9)\qquad \Gamma^h_{ij} = \left\{ {h \atop ij} \right\} + g_{ij}V^h - \delta^h_j V_i$$

and further, the formula (5) ascribes to

$$(10)\qquad R^k_{ihl} = G^k_{ihl} + g_{il}V^k_{;h} - g_{ih}V^k_{;l} - \delta^k_h V_{i;l} - \delta^k_l V_{i;h}$$
$$+ V_i(\delta^k_h V_l - \delta^k_l V_h)$$
$$+ (g_{il}V_h - g_{ih}V_l)V^k + (\delta^k_l g_{ih} - \delta^k_h g_{il})V^s V_s$$

where the notation $V_{k;h}$ and $V^k_{;h}$ indicate the absolute differentiation with respect to the fundamental tensor g_{ij} which grounds to base the Riemannian space without torsion and $G^k_{\ ihl}$ is the curvature tensor of the Riemannian space properly structured upon the metric (1). [2].

2. Some properties derived from the condition a). Contraction in (10) with respect to k and h leads

(11) $$R_{il}=G_{il}+g_{il}V^k_{\ ;k}(n-2)V_{i;l}+(n-2)V_iV_l-(n-2)g_{il}V^kV_k \quad (\neq R_{li}).$$

Again, contracting this (11) by g^{il}, we have

(12) $$R=G+2(n-1)V^k_{\ ;k}-(n-1)(n-2)V^kV_k.$$

Now here, we investigate the following several cases concerned with the number of dimensions.

Suppose that $n>3$, then we can deduce easily the following relations from (10), (11) and (12).

(13) $$\begin{aligned} &R^k_{\ ihl}-\frac{g_{il}R^k_{\ h}-g_{ih}R^k_{\ l}}{n-2}-\frac{\delta^k_hR_{il}-\delta^k_lR_{ih}}{n-2}+\frac{(\delta^k_hg_{il}-\delta^k_lg_{ih})R}{(n-1)(n-2)} \\ &=G^k_{\ ihl}-\frac{g_{il}G^k_{\ h}-g_{ih}G^k_{\ l}}{n-2}-\frac{\delta^k_hG_{il}-\delta^k_lG_{i}\delta}{n-2}+\frac{(\delta^k_hg_{il}-\delta^k_lg_{ih})G}{(n-1)(n-2)} \end{aligned}$$

From this, we obtain the following result:

THEOREM 1. *If $n>3$, Weyl's conformal curvature tensor in the pseudo-Riemannian space R^* with torsion vector is indifferent in the proper Riemannian space R.*

The case of $n=3$ is more clear, that is, the curvature tensor in R^* is identically zero, since the right hand side of (13) vanishes for $n=3$.

Now we put

(14) $$\begin{aligned} S_{il}&=2(n-1)R_{il}-g_{il}R, \\ H_{il}&=2(n-1)G_{il}-g_{il}G \end{aligned}$$

and remark that D_hS_{ij} is the absolute derivative with respect to the connection Γ^h_{ij}, but $S_{ij;h}$ denotes the covariant derivative with respect to $\{^{\,k}_{ij}\}$ and they are related with

$$D_hS_{il}=S_{il;h}-V^k(g_{lh}S_{ik}+g_{ih}S_{kl})-S_{ih}V_l+S_{hl}V_i.$$

Then we can deduce the following from (11) and (12)

(15) $$D_hS_{il}-D_lS_{ih}=H_{il;h}+H_{ih;l}+2(n-1)(n-2)W^k_{\ ihl}V_k$$

where $W^k_{\ ihl}$ represents the right hand side in the relation (13). According to the fact that $W^k_{\ ihl}=0$ for $n=3$, we have

(16) $$D_hS_{il}-D_lS_{ih}=H_{il;h}-H_{ih;l}.$$

Thus, we obtain the following.

THEOREM 2. *In the 3-dimensional space R^* with torsion, the tensor part of the conformal curvature is equal to the tensor part of the conformal curvature of proper Riemannian space R.*

Next, for the case of $n=2$ it is simple. The conditions (11) and (12) give us

(17) $$2R_{il}-g_{il}R=2G_{il}-g_{il}G=0.$$

That is, the Ricci tensors are same in both spaces. At the end, let us refer to the condition b) briefly. Since $R^k{}_{kih}=0$, it requires

$$V_i=\frac{\partial U}{\partial x^i}$$

and this means the torsion vector constitutes a field of gradient.

3. Some calculations and results. In the space R, we introduce the connection constructed as

(18) $$\Gamma^h{}_{ij}=\left\{ {h \atop ij} \right\}+g_{ij}V^h-\delta^h{}_jV_i$$

and for the geodesics

(19) $$\frac{d^2x^i}{ds^2}+\left\{ {i \atop kh} \right\}\frac{dx^k}{ds}\cdot\frac{dx^h}{ds}=0$$

and for the straight lines

(20) $$\frac{d^2x^i}{ds^2}+\left\{ {i \atop kh} \right\}\frac{dx^k}{ds}\cdot\frac{dx^h}{ds}+V^i-V_k\frac{dx^k}{ds}\cdot\frac{dx^i}{ds}=0$$

where the arc length

$$g_{ij}\frac{dx^i}{ds}\cdot\frac{dx^j}{ds}=1.$$

Of course, it is evident that our space does not always be able to admit the geodesics on surfaces, but possible on planes. For the sake of this, it is necessary the systems(see [3], [4])

(21) $$u_idx^i=0$$
$$du_i-u_k\Gamma^k_{ih}dx^h=u_i\omega+\lambda_iu_kdu^k$$

agree the complete integrability with Pfaffian form ω.

To be rewritten

$$u_idx^i=0$$

$$du^i - u_k \left\{ {k \atop ih} \right\} dx^h - u_k g_{ih} V^k dx^h + V_i u_h dx^h = u_i \gamma_h dx^h + \lambda_i u_h dx^h$$

or

(22) $$\begin{cases} u_i dx^i = 0 \\ du^i = [u_k \left\{ {k \atop ih} \right\} + g_{ih} u_k V^k + u_i \gamma_h + (\lambda_i - V_i) u_h] dx^h. \end{cases}$$

Here, the first condition of integrability gives

$$\gamma_i = \lambda_i - V_i.$$

Hence, (22) can be denoted as

(23) $$\begin{cases} u_i dx^i = 0 \\ u_{i;h} = \dfrac{\partial u_i}{\partial x^h} - u_k \left\{ {k \atop ih} \right\} = u_k V^k g_{ih} + u_i \gamma_h + u_h \gamma_i \end{cases}$$

and the second condition of integrability gives the relations

(24) $$\begin{aligned} G^k{}_{ihl} = & g_{ih}(V^k{}_{;l} + V^k V_l) - g_{il}(V^k{}_{;h} + V^k V_h) \\ & - \delta^k{}_l(\gamma_{i;h} - \gamma_i \gamma_h - g_{ih} V^s \gamma_s) \\ & + \delta^k{}_h(\gamma_{i;l} - \gamma_i \gamma_l - g_{il} V^s \gamma_s) \\ & - \delta^k{}_i(\gamma_{h;l} - \gamma_{l;h}). \end{aligned}$$

In this (24) remarking the properties $G^k{}_{khl} = 0$, we have the conditions

(25) $$V_i + (n+1)\gamma_i = \frac{\partial \varphi}{\partial x^i}$$

where φ be a function of x^i.

Further, contracting (24) with g^{ih}, we have

(26) $$\begin{aligned} -G^k{}_l = g^{ih} G^k{}_{ihl} = & (n-1)(V^k{}_{;l} + V^k V_l) \\ & + \gamma^k{}_{;l} - \gamma^k \gamma_l - \delta^k{}_l V_s \gamma^s \\ & - \delta^k{}_l(\gamma^s{}_{;s} - \gamma^s \gamma_s - n V^s \gamma_s) \\ & + g^{kh}(\gamma_{h;l} - \gamma_{l;h}) \end{aligned}$$

and by the way the covariant components are calculated as

(27) $$\begin{aligned} -G_{il} = & (n-1)(V_{i;l} + V_i V_l) \\ & + \gamma_{i;l} - \gamma_i \gamma_l - g_{il} V^s \gamma_s \\ & - g_{il}(\gamma^s{}_{;s} - \gamma^s \gamma_s - n V^s \gamma_s) \\ & + \gamma_{i;l} - \gamma_{l;i}. \end{aligned}$$

Then, the symmetry of G_{il} calls for

(28) $$(n-1) V_i + 3\gamma_i = \frac{\partial \phi}{\partial x^i}$$

where ϕ be a function of x^i.

For the case of $n > 2$, (25) and (28) are independent and hence we deduce

$$V_i=\frac{\partial U}{\partial x^i},\quad \gamma_{h;l}-\gamma_{l;h}=0.$$

Accordingly,

$$\gamma_i=\frac{\partial \gamma}{\partial x^i},\quad V_{h;l}-V_{l;h}=0.$$

Now getting together mentioned above, we can conclude:

THEOREM 3. *In the space with the asymmetric affine connection* (18), *geodesics* (19) *and straight lines* (20), *if* $n>2$ *then the torsion vector makes a field of gradient and the space is reduced to the pseudo-Riemannian.*

Again, the contraction in (26) leads

$$-G=(n-1)(V^s{}_{;s}+V^sV_s)-(n-1)(\gamma^s{}_{;s}-\gamma^s\gamma_s-nV^s\gamma_s) \tag{29}$$

and the other contraction in (24) with respect to k and l gives us

$$\begin{aligned}-G_{ih}=g_{ih}(V^s{}_{;s}+V^sV_s)-(V_{i;h}+V_iV_h)\\ -(n-1)(\gamma_{i;h}-\gamma_i\gamma_h-g_{ih}V^s\gamma_s).\end{aligned} \tag{30}$$

Thence, the mixed components are calculated as follows:

$$\begin{aligned}-G^k{}_l=\delta^k{}_l(V^s{}_{;s}+V^sV_s)-(V^k{}_{;l}+V^kV_l)\\ -(n-1)(\gamma^k{}_{;l}-\gamma^k\gamma_l-\delta^k{}_lV^s\gamma_s).\end{aligned} \tag{31}$$

Finally from above relations (26), (27), (28), (29), (30) and (31), we obtain

$$\begin{aligned}-\frac{\delta^k_lG_{ih}+g_{ih}G^k{}_l}{n-2}+\frac{\delta^k_lg_{ih}G}{(n-1)(n-2)}\\ =g_{ih}(V^k{}_{;l}+V^kV_l)-\delta^k_l(\gamma_{i;h}-\gamma_i\gamma_h-g_{ih}V^s\gamma_s).\end{aligned} \tag{32}$$

Accordingly, from (32) and (24), we have

$$G^k{}_{ihl}+\frac{g_{ih}G^k{}_l-g_{il}G^k{}_h+\delta^k_lG_{ih}-\delta^k_hG_{il}}{n-2}-\frac{(\delta^k_lg_{ih}-\delta^k_hg_{il})G}{(n-1)(n-2)}=0 \tag{33}$$

and further

$$n(V^k{}_{;l}+V^kV_l)-\delta^k_l(V^s{}_{;s}+V^sV_s)+n(\gamma^k{}_{;l}-\gamma^k\gamma_l)-\delta^k_l(\gamma^s{}_{;s}-\gamma^s\gamma_s)=0. \tag{34}$$

Forthwise we obtain the following relations which have to determine the V_i:

$$-nG_{ih}+g_{ih}G=n(n-2)(V_{i;h}+V_iV_h)-(n-2)g_{ih}(V^s{}_{;s}+V^sV_s). \tag{35}$$

Putting

$$V_{i;h}+V_iV_h=-\frac{G_{ih}}{n-2}+g_{ih}A, \tag{36}$$

$$V^s{}_{;s}+V^sV_s=-\frac{G}{n-2}+nA \tag{37}$$

then the condition of integrability of the first of above (36) gives us

$$(38)\qquad (n-2)\, V_k G^k{}_{ihl} - V_h G_{il} + V_l G_{ih} + G_{ih;l} - G_{il;h} = (n-2)\Big[g_{ih}\Big(\frac{\partial A}{\partial x^l} + A V_l\Big) - g_{il}\Big(\frac{\partial A}{\partial x^h} + A V_h\Big)\Big].$$

Contracting with g^{il}, we have

$$(39)\qquad (n-1)\, V_k G^k{}_h - G V_h = \frac{1}{2}\, \frac{\partial G}{\partial x^h} - (n-1)\Big(\frac{\partial A}{\partial x^h} + A V_h\Big).$$

That is, the relation (38) can be rewritten as

$$V_k\Big[G^k{}_{ihl} + \frac{\delta^k_l G_{il} - \delta^k_h G_{il}}{n-2} + \frac{g_{ih} G^k{}_l - g_{il} G^k{}_h}{n-2} - \frac{(\delta^k_l g_{ih} - \delta^k_h g_{il}) G}{(n-1)(n-2)}\Big] + \frac{(n-1)(G_{ih;l} - G_{il;h}) - g_{ih}\dfrac{\partial G}{\partial x^l} + g_{il}\dfrac{\partial G}{\partial x^k}}{2(n-1)(n-2)} = 0.$$

However, the first term of the left hand side of above should be vanished according to (33). Thus we have

$$(40)\qquad G_{ih;l} - G_{il;h} = \frac{1}{2(n-1)}\Big(g_{ih}\frac{\partial G}{\partial x^l} - g_{il}\frac{\partial G}{\partial x^h}\Big).$$

Thence, we have the following results:

THEOREM 4. *The condition such that the space (Theorem 3) with torsion admits planes (surfaces totally rectilinear) is that (33) and (40) are reducible to each other.*

THEOREM 5. *If $n>3$, the condition (40) follows from (33). If $n=3$, the relations (35) are identically zero and in any case of either (33) or (40), the Riemannian space is conformally equivalent with the ordinary Euclidian space, thus, its metric takes the from:*

$$ds^2 = e^{2\varphi} \sum (dx^i)^2$$

where φ be a function of x^i.

4. A note. The differential equations

$$\frac{d^2 x^i}{ds^2} + \left\{ {i \atop jk} \right\} \cdot \frac{dx^j}{ds} \cdot \frac{dx^k}{ds} + V^i - V_j \frac{dx^j}{ds} \cdot \frac{dx^i}{ds} = 0$$

of the straight lines of the space and the equations

$$(25)\qquad V_i + (n+1)\gamma_i = \frac{\partial \varphi}{\partial x^i}$$

of its planes are reciprocal. That is, the integral with $2(n-1)$ parameters of the straight lines and the integral with n parameters of the planes are just as that these planes totally contain the straight lines. Since then, it is sure that the differential systems of the straight lines and the planes are reducible to the differential systems:

$$\frac{d^2x^i}{ds^2}=0,$$

$$u_i dx^i=0, \quad \frac{\partial u_i}{\partial x^k}=0.$$

Putting $ds^2=e^{2\varphi}\sum(dx^i)^2$, we have

$$G_{ij}=-(n-2)\left(\frac{\partial^2\varphi}{\partial x^i\partial x^j}-\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}\right)-\delta_{ij}\left[\sum\frac{\partial^2\varphi}{(\partial x^k)^2}+(n-2)\sum\left(\frac{\partial\varphi}{\partial x^k}\right)^2\right].$$

The equations (35) immediately may be resolved and show $U=\varphi$ whence

$$V_i=\frac{\partial\varphi}{\partial x_i}$$

and the equations (34) show that putting $\gamma_i=-V_i$ is sufficient because γ_i are arbitrary whatever they must be represented in the form of the derivative $\frac{\partial\gamma}{\partial x^i}$.

The differential equations of the geodesics can be written down:

$$\frac{d^2x^i}{ds^2}+2\frac{\partial\varphi}{\partial x^k}\cdot\frac{dx^k}{ds}\cdot\frac{dx^i}{ds}-\frac{\partial\varphi}{\partial x^i}\sum\left(\frac{dx^k}{ds}\right)^2=0$$

and the differential equations of the straight lines are

$$\frac{d^2x^i}{ds^2}+\frac{\partial\varphi}{\partial x^k}\cdot\frac{dx^k}{ds}\cdot\frac{dx^i}{ds}-\frac{\partial\varphi}{\partial x^i}\sum\left(\frac{dx^k}{ds}\right)^2+\frac{\partial\varphi}{\partial x^i}e^{-2\varphi}=0.$$

Remarking that

$$e^{2\varphi}\sum\left(\frac{dx^k}{ds}\right)^2=1,$$

we can find

$$\frac{d^2x^i}{ds^2}+\frac{\partial\varphi}{\partial x^k}\cdot\frac{dx^k}{ds}\cdot\frac{dx^i}{ds}=0$$

and further

$$\frac{dx^i}{ds}=a^ie^{-\varphi}, \quad \sum(a^i)^2=1,$$

where

$$x^i=a^i\lambda+b^i,$$

$$d\lambda=\sqrt{\sum(dx^i)^2}.$$

This shows that our straight lines are the real straight lines of the ordinary Euclidean space like as we have already proved and they are whole parallel (absolutely parallel).

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