On Functors Text over Commutative Rings

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Let A, B and C be modules over a commutative ring R. If we take $X \to A$, $Y \to B$, $C \to I$ as projective resolutions over A, B and an injective resolution over C, respectively, then we get a complex $\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y,I))$ of R-modules. Here we can define a new functor Text_R from the category of all R-modules and homomorphisms to itself such that $\mathrm{Text}_R(A,B,C) = H\ (\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y,I))$, where H is the homology functor (see § 1).

In general it is difficult that we find some properties of Text and compute $\mathrm{Text}_R(A,B,C)$. In this paper we shall try to find some properties of Text_R and to compute $\mathrm{Text}_R(A,B,C)$ under some special conditions (see §§ 1-3 and Example 3). Finally, we shall prove some properties of Text using spectral sequences (see § 4).

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1. The definition of Text

Let K and L be complexes over a commutative ring R. We shall define a complex $\mathrm{Hom}_R(K,L)$ with lower indices as follows.

Set $\mathrm{Hom}_n(K,L) = \prod_{p=-\infty}^{\infty} \mathrm{Hom}_R(K_p, L_{n+p})$ so that an element f of $\mathrm{Hom}_n(K,L)$ is a family of homomorphisms $f_p : K_p \longrightarrow L_{n+p}$ for $-\infty < p < \infty$. When we assume that the boundaries in K and L are ∂_K and ∂_L the boundary ∂_H' in $\mathrm{Hom}_R(K,L)$ is defined by

$$(\partial_H'f)_p(k_p) = \partial_L(f_p k_p) + (-1)^{n+1} f_{p-1}(\partial_K k_p) \text{ and } \partial_H' f = \text{the family of } (\partial_H'f)_p \qquad (1)$$

for $k_p \in K_p$ and $f_p, f_{p-1} \in f$. (Note: Consider an element $f = \{f_p \mid f_p : K_p \longrightarrow L_{n+p}\}$ such that for each $k_m \in K_m$, $f_m k_m = 0$ if $m \neq p$ in $\mathrm{Hom}_n(K, L)$. Then we see that

$$(\partial_H'f)_{p+1}(k_{p+1}) = (-1)^{n+1} f_p(\partial_K k_{p+1}),\ (\partial_H'f)_p(k_p) = \partial_L(f_p k_p)$$
$$\text{and } (\partial_H'f)_m(k_m) = 0 \text{ if } m \neq p \text{ and } p+1.)$$

We know $\partial_H'\partial_H' = 0$ by the calculation:

$$\begin{aligned}(\partial_H'\partial_H'f)_p(k_p) &= \partial_L((\partial_H'f)_p(k_p)) + (-1)^n(\partial_H'f)_{p-1}(\partial_K k_p) \\ &= \partial_L(\partial_L(f_p k_p) + (-1)^{n+1} f_{p-1}(\partial_K k_p)) \\ &\quad + (-1)^n \partial_L f_{p-1}(\partial_K k_p) + (-1)^{2n+1} f_{p-1}(\partial_K \partial_K k_p)\end{aligned}$$

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$$= \partial_L \partial_L (f_p k_p) + (-1)^{n+1} \partial_L (f_{p-1}(\partial_K k_p))$$
$$+ (-1)^n \partial_L (f_{p-1}(\partial_K k_p)) + (-1)^1 f_{p-2}(\partial_K \partial_K k_p) = 0,$$

where $k_p \in K_p$ (see page 43 of [2]).

We shall add a complex M over R(commutative ring) with the boundary ∂_M in the above situation, then we get the complex $\mathrm{Hom}_R(K, \mathrm{Hom}_R(L, M))$ with the boundary ∂_H such that

$$\begin{aligned} \mathrm{Hom}_n(K, \mathrm{Hom}_R(L, M)) &= \prod_{p=-\infty}^{\infty} \mathrm{Hom}_R(K_p, \mathrm{Hom}_{n+p}(L, M)) \\ &= \prod_{p=-\infty}^{\infty} \mathrm{Hom}_R(K_p, \prod_{q=-\infty}^{\infty} \mathrm{Hom}_R(L_q, M_{n+p+q})) \\ &= \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Hom}_R(K_p, \mathrm{Hom}_R(L_q, M_{n+p+q})). \end{aligned}$$

$$\begin{aligned} ((\partial_H f)_p k_p)_q (l_q) &= \partial_H'((f_p k_p)_q (l_q)) + (-1)^{n+1} (f_{p-1}(\partial_K k_p))_q (l_q) \\ &= \partial_M ((f_p k_p)_q (l_p)) + (-1)^{n+p+1} (f_p k_p)_{q-1} (\partial_L l_q) \\ &\quad + (-1)^{n+1} f_{p-1}(\partial_K k_p))_q (l_q) \text{ (see (1))} \end{aligned} \tag{2}$$

for $k_p \in K_p$, $l_q \in L_q$, $f_p : K_p \longrightarrow \mathrm{Hom}_{n+p}(L, M)$, $(f_p k_p)_q : L_q \longrightarrow M_{n+p+q}$, and so on, where ∂_H' is the boundary in $\mathrm{Hom}_R(L, M)$.

With the above situation we also define

$$\mathrm{Hom}_n(K \otimes_R L, M) = \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Hom}_R(K_p \otimes_R L_q, M_{n+p+q}),$$

$$\begin{aligned} (\bar{\partial}_H \bar{f})_{p,q}(k_p \otimes l_q) &= \partial_M (\bar{f}_{p,q}(k_p \otimes l_q)) + (-1)^{n+1} \bar{f}_{p-1,q}(\partial_K k_p \otimes l_q) \\ &\quad + (-1)^{n+p+1} \bar{f}_{p,q-1}(k_p \otimes \partial_L l_q) \end{aligned} \tag{3}$$

for $k_p \in K_p$, $l_q \in L_q$, $\bar{f} \in \mathrm{Hom}_n(K \otimes_R L, M)$, $\bar{f}_{p-1,q} : K_{p-1} \otimes_R L_q \longrightarrow M_{n+p+q-1}$ in $\bar{f}$, and so on, where $\bar{\partial}_H$ is the boundary in $\mathrm{Hom}_R(K \otimes_R L, M)$ and the complex $K \otimes_R L$ is defined by $(K \otimes_R L)_n = \sum_{p+q=n} (K_p \otimes_R L_q)$ with the boundary $\bar{\partial}' \otimes (k_p \otimes l_q) = \partial_K k_p \otimes l_q + (-1)^p k_p \otimes \partial_L l_q$.

Using the natural isomorphism $\eta : \mathrm{Hom}_R(K, \mathrm{Hom}_R(L, M)) \cong \mathrm{Hom}_R(K \otimes_R L, M)$ we can prove $\eta(\partial_H f) = \bar{\partial}_H(\eta f)$, where $f \in \mathrm{Hom}_n(K, \mathrm{Hom}_R(L, M))$. By (2) and (3) we have

$$\begin{aligned} (\bar{\partial}_H(\eta f))_{p,q}(k_p \otimes l_q) &= \partial_M (\eta f)_{p,q}(k_p \otimes l_q)) + (-1)^{n+1} (\eta f)_{p-1,q}(\partial_K k_p \otimes l_q) \\ &\quad + (-1)^{n+p+1} (\eta f)_{p,q-1}(k_p \otimes \partial_L l_q) \\ &= \partial_M ((f_p k_p)_q (l_q)) + (-1)^{n+1} (f_{p-1}(\partial_K k_p))_q (l_q) \\ &\quad + (-1)^{n+p+1} (f_p k_p)_{q-1} (\partial_L l_q) \\ &= ((\partial_H f)_p (k_p))_q (l_q) = (\eta(\partial_H f))_{p,q}(k_p \otimes l_q). \end{aligned}$$

where $(\eta f)_{p,q}(k_p \otimes l_q) = (f_p k_p)_q (l_q)$ by the definition of η (see page 144 of [2]). Since $(\bar{\partial}_H(\eta f))_{p,q}(k_p \otimes l_q) = (\eta(\partial_H f))_{p,q}(k_p \otimes l_q)$ is true for all p, q, and n as above we have

$$H_n(\mathrm{Hom}_R(K, \mathrm{Hom}_R(L, M))) \cong H_n(\mathrm{Hom}_R(K \otimes_R L, M)). \tag{4}$$

As a special case we shall take a situation which (a) K and L both are positive complexes with lower indices such that

$$K: \cdots\cdots \longrightarrow K_m \xrightarrow{\partial_K} K_{m-1} \longrightarrow \cdots\cdots \longrightarrow K_0 \longrightarrow 0$$

$$L: \cdots\cdots \longrightarrow L_m \xrightarrow{\partial_L} L_{m-1} \longrightarrow \cdots\cdots \longrightarrow L_0 \longrightarrow 0$$

(b) M is a negative complex with lower indices such that

$$M: 0 \longrightarrow M_0 \longrightarrow M_{-1} \longrightarrow \cdots\cdots \longrightarrow M_{-m} \xrightarrow{\partial_M} M_{-m-1} \longrightarrow \cdots\cdots$$

then $\mathrm{Hom}_R(K, \mathrm{Hom}_R(L, M))$ becomes a negative complex with lower indices because n in $\mathrm{Hom}_n(K, \mathrm{Hom}_R(L, M))$ should be zero or minus to preserve zero or minus indices in M. We shall write down Hom_n $(K, \mathrm{Hom}_R(L, M))$ of this case in detail:

$$\mathrm{Hom}_0(K, \mathrm{Hom}_R(L, M)) = \mathrm{Hom}_R(K_0, \mathrm{Hom}_0(L, M)) = \mathrm{Hom}_R(K_0, \mathrm{Hom}_R(L_0, M_0))$$
$$\mathrm{Hom}_{-1}(K, \mathrm{Hom}_R(L, M)) = \mathrm{Hom}_R(K_0, \mathrm{Hom}_R(L_1, M_0))$$
$$\oplus \mathrm{Hom}_R(K_0, \mathrm{Hom}_R(L_0, M_{-1})) \oplus \mathrm{Hom}_R(K_1, \mathrm{Hom}_R(L_0, M_0)), \cdots\cdots$$

Therefore if we put $\mathrm{Hom}_{-n} = \mathrm{Hom}^n$, $M_{-n} = M^n$ then $\mathrm{Hom}^n(K, \mathrm{Hom}_R(L, M))$ is expressed as follows:

$$\mathrm{Hom}^0(K, \mathrm{Hom}_R(L, M)) = \mathrm{Hom}_R(K_0, \mathrm{Hom}_R(L_0, M^0))$$
$$\mathrm{Hom}^1(K, \mathrm{Hom}_R(L, M)) = \mathrm{Hom}_R(K_0, \mathrm{Hom}_R(L_1, M^0))$$
$$\oplus \mathrm{Hom}_R(K_0, \mathrm{Hom}_R(L_0, M^1)) \oplus \mathrm{Hom}_R(K_1, \mathrm{Hom}_R(L_0, M^0)), \cdots\cdots$$
$$\mathrm{Hom}^n(K, \mathrm{Hom}_R(L, M)) = \sum_{p=0}^{n} \sum_{q=0}^{n-p} \mathrm{Hom}_R(K_p, \mathrm{Hom}_R(L_q, M^{n-p-q})), \cdots\cdots$$

With the above preparation we shall define the functor Text. Let A, B and C be modules over a commutative ring R. Take $\cdots\cdots \longrightarrow X_n \xrightarrow{\partial_A} X_{n-1} \longrightarrow \cdots\cdots \longrightarrow X_0 \xrightarrow{\varepsilon_A} A \longrightarrow 0$ as a projective resolution over A, $\cdots\cdots \xrightarrow{\partial_B} Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots\cdots \longrightarrow Y_0 \xrightarrow{\varepsilon_B} B \longrightarrow 0$ as a projective resolution over B and $0 \longrightarrow C \xrightarrow{\varepsilon_C} I^0 \longrightarrow \cdots\cdots \longrightarrow I^n \xrightarrow{\partial_C} I^{n+1} \longrightarrow \cdots\cdots$ as an injective resolution over C. We then get the complex $\mathrm{Hom}^n(X, \mathrm{Hom}_R)Y, I))$ $= \sum_{p=0}^{n} \sum_{q=0}^{n-p} \mathrm{Hom}_R(X_p, \mathrm{Hom}_R(Y_q, I^{n-p-q}))$ $(n \geq 0)$ with boundary ∂_H such that

$$((\partial_H f)_p(x_p))_q(y_q) = \partial_C((f_p x_p))_q(y_q)) + (-1)^{n-p+1}(f_p x_p)_{q-1}(\partial_B y_q)$$
$$+ (-1)^{n+1}(f_{p-1}(\partial_A x_p))_q(y_q)$$

as (2), where $x_p \in X_p$ and $y_q \in Y_q$. Define

$$\mathrm{Text}_R^n(A, B, C) = H^n(\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I)) \quad (n \geq 0)$$

$$(\cong H^n(\mathrm{Hom}_R(X \otimes_R Y, I)) \quad \text{by (4) } n \geq 0),$$

where H is the homology functor for ∂_H (for $\bar{\partial}_H$, see (3)). We shall prove that $\mathrm{Text}_R^n(A, B, C)\,(n \geq 0)$ is independent of the choice of X, Y and I.

Let us take other projective resolutiens $X' \xrightarrow{\varepsilon_A'} A$ with the boundary ∂_A' and $Y \xrightarrow{\varepsilon_B'} B$ with the boundary ∂_B' and another injective resolution $C \xrightarrow{\varepsilon_C'} I'$ with the boundary ∂_C'. Then there are chain transformations φ and φ' in the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon_A} & A \\ \varphi_A \downarrow \uparrow \varphi_A' & & \| \\ X' & \xrightarrow{\varepsilon_A'} & A \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\varepsilon_B} & B \\ \varphi_B \downarrow \uparrow \varphi_B' & & \| \\ Y' & \xrightarrow{\varepsilon_B'} & B \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\varepsilon_C} & I \\ \| & & \varphi_C \downarrow \uparrow \varphi_C' \\ C & \xrightarrow{\varepsilon_C'} & I' \end{array} \qquad (5)$$

satisfying

$$\begin{pmatrix} \varphi_A'\varphi_A \simeq 1_X \\ \varphi_A\varphi_A' \simeq 1_{X'} \end{pmatrix} \qquad \begin{pmatrix} \varphi_B'\varphi_B \simeq 1_Y \\ \varphi_B\varphi_B' \simeq 1_{Y'} \end{pmatrix} \text{ and } \qquad \begin{pmatrix} \varphi_C'\varphi_C \simeq 1_I \\ \varphi_C\varphi_C' \simeq 1_{I'} \end{pmatrix}$$

where $\simeq$ means that both sides are chain homotopic. There is then the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(B, C) & \xrightarrow{\mathrm{Hom}(\varepsilon_B, \varepsilon_C')} & \mathrm{Hom}_R(Y, I) \\ \| & & \mathrm{Hom}(\varphi_B', \varphi_C') \uparrow \; \downarrow \mathrm{Hom}(\varphi_B', \varphi_C) \\ \mathrm{Hom}_R(B, C) & \xrightarrow{\mathrm{Hom}(\varepsilon_B', \varepsilon_C')} & \mathrm{Hom}_R(Y', I') \end{array} \qquad (6)$$

satisfying $\quad \mathrm{Hom}(\varphi_B, \varphi_C') \cdot \mathrm{Hom}(\varphi_B', \varphi_C) = \mathrm{Hom}(\varphi_B'\varphi_B, \varphi_C'\varphi_C) \simeq 1_{\mathrm{Hom}_R}(Y, I)$

$\quad \mathrm{Hom}(\varphi_B', \varphi_C) \cdot \mathrm{Hom}(\varphi_B', \varphi_C') = \mathrm{Hom}(\varphi_B\varphi_B', \varphi_C\varphi_C') \simeq 1_{\mathrm{Hom}_R}(Y', I')$

where $\mathrm{Hom}(\varphi_B'. \varphi_C)$ and $\mathrm{Hom}(\varphi_B, \varphi_C')$ are chain transformations which implies $H(\mathrm{Hom}_R(Y, I)) \cong H(\mathrm{Hom}_R(Y', I'))$.

From (5) and (6) we also get the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)) & \xrightarrow{\mathrm{Hom}(\varepsilon_A, \mathrm{Hom}(\varepsilon_B, \varepsilon_C))} & \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I)) \\ \| & & \mathrm{Hom}(\varphi_A, \mathrm{Hom}(\varphi_B, \varphi_C')) \uparrow \; \downarrow \mathrm{Hom}(\varphi_A', \mathrm{Hom}(\varphi_B', \varphi_C)) \\ \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)) & \xrightarrow{\mathrm{Hom}(\varepsilon_A', \mathrm{Hom}(\varepsilon_B', \varepsilon_C'))} & \mathrm{Hom}_R(X', \mathrm{Hom}_R(Y', I')) \end{array}$$

satisfying

$$\mathrm{Hom}(\varphi_A, \mathrm{Hom}(\varphi_B, \varphi_C')) \cdot \mathrm{Hom}(\varphi_A', \mathrm{Hom}(\varphi_B', \varphi_C))$$
$$= \mathrm{Hom}(\varphi_A'\varphi_A, \mathrm{Hom}(\varphi_B\varphi_B', \varphi_C'\varphi_C)) \simeq 1_{\mathrm{Hom}_R}(X, I))$$
$$\mathrm{Hom}(\varphi_A', \mathrm{Hom}(\varphi_B', \varphi_C)) \cdot \mathrm{Hom}(\varphi_A, \mathrm{Hom}(\varphi_B, \varphi_C'))$$
$$= \mathrm{Hom}(\varphi_A\varphi_A', \mathrm{Hom}(\varphi_B\varphi_B', \varphi_C\varphi_C')) \simeq 1_{\mathrm{Hom}_R}(X', \mathrm{Hom}_R(Y', I'))$$

where $\mathrm{Hom}(\varphi_A, \mathrm{Hom}(\varphi_B, \varphi_C'))$ and $\mathrm{Hom}(\varphi_A', \mathrm{Hom}(\varphi_B', \varphi_C))$ are chain transformations.

This implieis

$$H(\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I))) \cong H(\mathrm{Hom}_R(X', \mathrm{Hom}_R(Y', I'))).$$

We shall prove that $\mathrm{Text}_R(A, B, C) \cong \mathrm{Text}_R(B, A, C)$. Since

$$\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)) \cong \mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(B \otimes_R A, C)$$
$$\cong \mathrm{Hom}_R(B, \mathrm{Hom}_R(A, C)),$$ in consequence we have that

$$H^n(\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I)) \cong H^n(\mathrm{Hom}_R(X \otimes_R Y, I)) \cong H^n(\mathrm{Hom}_R(Y \otimes_R X, I)$$
$$\cong H^n(\mathrm{Hom}_R(Y, \mathrm{Hom}_R(X, I))$$

which implies $\mathrm{Text}_R^n(A, B, C) \cong \mathrm{Text}_R^n(B, A, C)$, where X, Y and I are the same one as in the definition of Text.

EXAMPLE 1. If $\mathrm{Tor}_n^R(B, C) = 0$ for $n \geq 1$ then for projective resolutions $X' \longrightarrow B$ and $X'' \longrightarrow C$ over R-modules B and C, respectively, $X' \otimes_R X''$ is a projective resolution over $B \otimes_R C$. Let us take a projective resolution $X \longrightarrow A$ over a R-moiule A and an injective resolution $D \longrightarrow I$ over a R-module D. We have then

$$H^n(\mathrm{Hom}_R(X \otimes_R X' \otimes_R X'', I) \cong H^n(\mathrm{Hom}_R(X, \mathrm{Hom}_R(X' \otimes_R X'', I)))$$
$$= \mathrm{Text}_R^n(A, B \otimes_R C, D)$$

If we put the right derived functor of $\mathrm{Hom}_R(A \otimes_R B \otimes_R C, D) = \mathrm{Quext}_R$ then we have $\mathrm{Text}_R^n(A, B \otimes_R C, D) \cong \mathrm{Quext}_R^n(A, B, C, D)$ under the condition $\mathrm{Tor}_n^R(B, C) = 0$ for $n \geq 1$.

In conseqence, Text_R in the right derived functor of $\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$ and contravariant in A, B, and covariant in C.

LEMMA 1. $\mathrm{Text}_R^0(A, B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$.

Proof. In the sequence

$$0 \longrightarrow \mathrm{Hom}_R(X_0, \mathrm{Hom}_R(Y_0, I^0)) \xrightarrow{\partial_H'} \mathrm{Hom}_R(X_1, \mathrm{Hom}_R(Y_0, I^0))$$
$$\oplus \mathrm{Hom}_R(X_0, \mathrm{Hom}_R(Y_1, I^0))$$
$$\oplus \mathrm{Hom}_R(X_0, \mathrm{Hom}_R(Y_0, I^1)),$$

$\mathrm{Ker}\, \partial_H^1 = \mathrm{Text}_R^0(A, B, C)$. Since there are two exact sequences

$$X_1 \xrightarrow{\partial_A} X_0 \xrightarrow{\varepsilon_A} A \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}_R(B, C) \xrightarrow{\mathrm{Hom}(\varepsilon_B, \varepsilon_C)} \mathrm{Hom}_R(Y_0, I^0) \xrightarrow{\partial'_H} \mathrm{Hom}_R(Y_0, I^1) \oplus \mathrm{Hom}_R(Y_1, I^0)$$

and Hom is left exact in each argument we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)) \longrightarrow \mathrm{Hom}_R(X_0, \mathrm{Hom}_R(Y_0, I^0)) \xrightarrow{\mathrm{Hom}(1, \partial_H') + \mathrm{Hom}(\partial_A, 1)}$$
$$\mathrm{Hom}_R(X_1, \mathrm{Hom}_R(Y_0, I^0)) \oplus \mathrm{Hom}_R(X_0, \mathrm{Hom}_R(Y_1, I^0)) \oplus \mathrm{Hom}_R(X_0, \mathrm{Hom}_R(Y_0, I^1))$$ (see Proposition 4.3a on page 25 of [1]). Since $\partial_H^1 = \mathrm{Hom}(1, \partial_H') - \mathrm{Hom}(\partial_A, 1)$ we have

Ker $\partial_H^1 = \mathrm{Text}_R^0(A, B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$.

We can easily derive the following.

(i) From Lemma 1 above Text_R^0 is left exact in each argument.

(ii) If A is projective, then $\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I))$ becomes

$$0 \longrightarrow \mathrm{Hom}^0(A, \mathrm{Hom}_R(Y, I)) \xrightarrow{\partial_H} \mathrm{Hom}^1(A, \mathrm{Hom}_R(Y, I)) \longrightarrow \cdots\cdots\cdots$$

In general, since $0 \longrightarrow \mathrm{Hom}^0(Y, I) \xrightarrow{\partial_H'} \mathrm{Hom}^1(Y, I) \longrightarrow \cdots\cdots\cdots$ is not exact Text_R^n $(A, B, C) \neq 0$ for $n \geq 0$. This is true when B (or C) is projective (or injective). (see Corollary 1 in § 2.)

(iii) If A and B are projective then $\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I))$ becomes

$$0 \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, I^0)) \xrightarrow{\partial_H} \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, I^1)) \longrightarrow \cdots\cdots\cdots$$

which is exact. Therefore $\mathrm{Text}_R^n(A, B, C) = 0$ for $n \geq 1$. This is also true when A (or B) is projective and C is injective.

(iv) For an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of R-modules we can always take projective resolutions X', X and X'' over A', A and A'', respectively, such that $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is split exact (see page 79 of [1]). We ahve therefore the exact sequence $0 \longrightarrow \mathrm{Hom}_R(X'', \mathrm{Hom}_R(Y, I)) \longrightarrow \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I)) \longrightarrow \mathrm{Hom}_R(X', \mathrm{Hom}_R(Y, I)) \longrightarrow 0$ where $Y \longrightarrow B$ is a projective resolution over the R-module B and $C \longrightarrow I$ is an injective resolution over the R-module C. Therefore there is the long exact sequence

$$0 \longrightarrow \mathrm{Text}_R^0(A'',B,C) \longrightarrow \mathrm{Text}_R^0(A,B,C) \longrightarrow \mathrm{Text}_R^0(A', B, C) \longrightarrow \mathrm{Text}_R^1(A'', B, C) \longrightarrow \cdots.$$

2. Speical Cases

Let K and L be complexes over a commutative ring R with boundaries ∂_K and ∂_L, respectively. To prove Theorem 1 below we shall show the following.

LEMMA 2. *If every K_p in K is projective as a R-module and the boundary ∂_K in K is identically zero, then there is an isomorphism*

$$\alpha_n : H_n(\mathrm{Hom}_R(K, L)) \cong \prod_{p=-\infty}^{\infty} \mathrm{Hom}_R(K_p, H_{n+p}(L)).$$

Proof. Put $\partial_L(L_{n+p+1}) = \mathrm{Im}(\partial_L)_{n+p}$, the kernel of the map $\partial_L : L_{n+p} \longrightarrow L_{n+p-1} = \mathrm{Ker}(\partial_L)_{n+p}$, and so on. We have then the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Im}(\partial_L)_{n+p} & \longrightarrow & \mathrm{Ker}(\partial_L)_{n+p} & \longrightarrow & H_{n+p}(L) & \longrightarrow & 0 \\ & & \partial_L \uparrow & & \downarrow & & & & \\ & & L_{n+p+1} & \xrightarrow{\partial_L} & L_{n+p} & & & & \\ & & & & \downarrow \partial_L & & & & \\ & & & & L_{n+p-1} & & & & \end{array}$$

where each row and column is exact. Now, since each K_p is projective the functor $\mathrm{Hom}_R(K_p, -)$ is exact. From these facts we have therefore the commutative diegram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(K_p, \mathrm{Im}(\partial_L)_{n+p}) & \longrightarrow & \mathrm{Hom}_R(K_p, \mathrm{Ker}(\partial_L)_{n+p}) & \longrightarrow & \mathrm{Hom}_R(K_p, H_{n+p}(L)) & \longrightarrow & 0 \\ & & \uparrow & & \downarrow & & & & \\ & & \mathrm{Hom}_R(K_p, L_{n+p+1}) & \xrightarrow{*} & \mathrm{Hom}_R(K_p, L_{n+p}) & & & & \\ & & & & \downarrow * & & & & \\ & & & & \mathrm{Hom}_R(K_p, L_{n+p-1}) & & & & \end{array}$$

with each row and column exact, where the arrows with * are the boundary ∂_H' in $\mathrm{Hom}_R(K, L)$ $(\partial_K = 0)$. This implies that for each p

$$\mathrm{Hom}_R(K_p, H_{n+p}(L)) = \text{the } p\text{-coordinate of } H_n(\mathrm{Hom}_R(K, L)).$$

We then proved our lemma.

As in § 1, let A, B and C be R-modules and their projective or injective resolutions with boundaries ∂_A, ∂_B, ∂_C be $X \longrightarrow A$, $Y \longrightarrow B$ and $C \longrightarrow I$, respectively. Set

image of $\partial_A = \mathrm{Im}(X)$, *i.e.*, image of ∂_A into $X_n = \mathrm{Im}(X)_n$
kernel of $\partial_A = \mathrm{Ker}(X)$, *i.e.*, kernel of ∂_A into $X_{n-1} = \mathrm{Ker}(X)_n$
cokernel of $\partial_A = \mathrm{Cok}(X)$, *i.e.*, cokernel of ∂_A in $X_n = \mathrm{Cok}(X)_n$
coimage of $\partial_A = \mathrm{Coim}(X)$, *i.e.*, coimage of ∂_A in $X_n = \mathrm{Coim}(X)_n$

and so on. We have the following as special cases.

THEOREM 1. *If* $X \to A$ *is* $0 \to X_1 \xrightarrow{\partial_A} X_0 \to A \to 0$ *(exact) there is an exact sequence* $0 \longrightarrow \mathrm{Ext}_R^1(A, \mathrm{Ext}_R^{n-1}(B, C)) \longrightarrow \mathrm{Text}_R^n(A, B, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Ext}_R^n(B, C)) \longrightarrow 0.$
If all quotients of each module in $\mathrm{Hom}_R(Y, I)$ *are injective then the above sequence splits.*

Proof. By the assumption we get

$$\begin{array}{lll} \mathrm{Im}(X)_0 \cong X_1, & \mathrm{Ker}(X)_0 = X_0, & \mathrm{Coim}(X)_0 = 0, \\ \mathrm{Im}(X)_1 = 0, & \mathrm{Ker}(X)_1 = 0, & \mathrm{Coim}(X)_1 = X_1. \end{array}$$

Therefore there are split exact sequences of complexes

$$0 \longrightarrow \mathrm{Ker}(X) \xrightarrow{i} X \underset{\varphi}{\overset{j}{\rightleftarrows}} \mathrm{Coim}(X) \longrightarrow 0 \tag{1}$$

and splitting homomorphisms φ. (Note: $\mathrm{Coim}(X)$ is a projective complex.) Moreover, we also get the exact sequences of complexes

$$E:\ 0 \longrightarrow \mathrm{Hom}_R(\mathrm{Coim}(X),\ \mathrm{Hom}_R(Y,\ I)) \xrightarrow{j^*} \mathrm{Hom}_R(X,\ \mathrm{Hom}_R(Y,\ I))$$
$$\xrightarrow{i^*} \mathrm{Hom}_R(\mathrm{Ker}(X),\ \mathrm{Hom}_R(Y,\ I)) \longrightarrow 0$$

and the exact homology sequence of E

$$\cdots\cdots\xrightarrow{\partial_E^{n-1}} H^n(\mathrm{Hom}_R(\mathrm{Coim}(X),\ \mathrm{Hom}_R(Y,\ I))) \xrightarrow{j^*} H^n(\mathrm{Hom}_R(X,\ \mathrm{Hom}_R^n(Y,\ I)))$$
$$\xrightarrow{i^*} H^n(\mathrm{Hom}_R(\mathrm{Ker}(X),\ \mathrm{Hom}_R(Y,\ I))) \xrightarrow{\partial_E^n} \cdots\cdots,$$

where ∂_E^{n-1} and ∂_E^n are connecting homomorphisms. The middle portion of the above sequence can be expressed in terms of ∂_E as a short exact sequence

$$0 \longrightarrow \mathrm{Coker}\ \partial_E^{n-1} \longrightarrow \mathrm{Text}_R^n(A,\ B,\ C) \longrightarrow \mathrm{Ker} \longrightarrow \partial_E^n \longrightarrow 0. \tag{2}$$

For each p the sequences

$$S:\ 0 \longrightarrow \mathrm{Coim}(X)_{p+1} \xrightarrow{\partial_A'} \mathrm{Ker}(X)_p \longrightarrow H_p(X) \longrightarrow 0 \tag{3}$$

is exact and the homomorphism

$$\partial_A'^*:\ \mathrm{Hom}^n(\mathrm{Ker}(X),\ \mathrm{Hom}_R(Y,\ I)) \longrightarrow \mathrm{Hom}^{n+1}(\mathrm{Coim}(X),\ \mathrm{Hom}_R(Y,\ I))$$

is induced by (3), where ∂_A' is from the boundary ∂_A in X. In consequence the homomorphisms on homology induced by $\partial_A'^*$(up to sign) are connecting homomorphisms ∂_E. In detail, ∂_E is defined on cycles by the "switchback"(see page 45 of [2]) $j^{*-1}\partial_H i^{*-1}$, where ∂_H is the boundary in $\mathrm{Hom}_R(X,\ \mathrm{Hom}_R(Y,\ I))$ as before. Since $\mathrm{Ker}(X)$ has zero-homomorphisms as its boundary a cycle g in $\mathrm{Hom}^n(\mathrm{Ker}(X),\ \mathrm{Hom}(Y,\ I))$ is a family $\{g_p : \mathrm{Ker}(X)_p \longrightarrow \mathrm{Hom}^{n-p}(Y,\ I)\}$ with $\partial_H'g = 0$, where ∂_H' is the boundary in $\mathrm{Hom}_R(Y,\ I)$. In (1) we get $X_p \cong \mathrm{Ker}(X)_p \oplus \mathrm{Coim}(X)_p$ and hence each g_p can be extended to $f_p : X_p \longrightarrow \mathrm{Hom}^{n-p}(Y,\ I)$ with $\partial_H' f_p = 0$. That is, a cycle g in $\mathrm{Hom}^n(\mathrm{Ker}(X),\ \mathrm{Hom}_R(Y,\ I))$ can be extended to f in $\mathrm{Hom}^n(X,\ \mathrm{Hom}_R(Y,\ I))$ with $\partial_H' f = 0$ and $\partial_H f = \pm\partial_A^* f$ for this homomorphism f. Since $\partial_A : X_n \longrightarrow X_{n-1}$ is decomposed as $X_n \longrightarrow \mathrm{Coim}(X)_n \xrightarrow{\partial_A'} \mathrm{Ker}(X)_{n-1} \longrightarrow X_{n-1}$ we have $\partial_H f = \pm j^*\partial_A'^* i^* f$ for each f as above, where $\partial_A^* = j^*\partial_A'^* i^*$ and $\partial_H f = \pm\partial_A^* f$. If we take $i^{*-1} g$ to be f then $j^{*-1}\partial_H i^{*-1} g = \pm\partial_A'^* g$ since $i^* f = fi = g$. Therefore ∂_E is induced by $\pm\partial_A'^*$.

Using Lemma 2 and $\partial_E = \pm\partial_A'^*$ above we have the commutative diagram (up to sign)

$$\begin{array}{ccc} H^n(\mathrm{Hom}_R(\mathrm{Ker}(X),\ \mathrm{Hom}_R(Y,\ I)) & \xrightarrow{\partial_E = \pm\, \partial_A'^*} & H^{n+1}(\mathrm{Hom}_R(\mathrm{Coim}(X),\ \mathrm{Hom}_R(Y,\ I))) \\ \alpha_n \downarrow \wr\wr & & \alpha_{n+1} \downarrow \wr\wr \\ \prod_{p=-\infty}^{\infty} \mathrm{Hom}_R(\mathrm{Ker}(X)_p,\ H^{n-p}(\mathrm{Hom}_R(Y,\ I))) & \xrightarrow{\partial_A'^*} & \prod_{p=-\infty}^{\infty} \mathrm{Hom}_R(\mathrm{Coim}(X)_{p+1},\ H^{n-p}(\mathrm{Hom}_R(Y,\ I))). \end{array}$$

Hence $\mathrm{Ker}\ \partial_E \cong \mathrm{Ker}\ \partial_E'^*$(lower line) and $\mathrm{Coker}\ \partial_E' \cong \mathrm{Coker}\ \partial_A'^*$(lower line). On the other hand, from (3) we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(H_p(X),\ H^{n-p}(\mathrm{Hom}_R(Y,\ I))) \longrightarrow \mathrm{Hom}_R(\mathrm{Ker}(X)_p,\ H^{n-p}(\mathrm{Hom}_R(Y,\ I))) \xrightarrow{\partial_A'^*}$$
$$\mathrm{Hom}_R(\mathrm{Coim}(X)_{p+1},\ H^{n-p}(\mathrm{Hom}_R(Y,\ I))) \xrightarrow{S^*} \mathrm{Ext}^1_R(H_p(X),\ H^{n-p}(\mathrm{Hom}_R(Y,\ I))) \longrightarrow 0 \qquad (4)$$

which gives the kernels and cokernels of $\partial_A'^*$ as

$$\mathrm{Ker}\ \partial^n_E \cong \mathrm{Ker}\ \partial_A'^* \cong \prod_{p=-\infty}^{\infty} \mathrm{Hom}_R(H_p(X),\ H^{n-p}(\mathrm{Hom}_R(Y,\ I))) = \mathrm{Hom}_R(A,\ \mathrm{Ext}^n_R(B,\ C))$$
$$\mathrm{Coker}\ \partial^{n-1}_E \cong \mathrm{Coker}\ \partial_A'^* \cong \prod_{p=-\infty}^{\infty} \mathrm{Ext}^1_R(H_p(X),\ H^{n-p-1}(\mathrm{Hom}_R(Y,\ I))$$
$$= \mathrm{Ext}_R{}^1(A,\ \mathrm{Ext}^{n-1}_R(B,\ C)),$$

where we should note that $H_0(X) \cong A$, $H_p(X) = 0$ if $p \neq 0$ and $\mathrm{Ext}^1_R(\mathrm{Ker}(X)_p,\ H^{n-p}(\mathrm{Hom}_R(Y,\ I))) = 0$ ($\mathrm{Ker}\ (X)_p$ is projective). Hence we have the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_R(A,\ \mathrm{Ext}^{n-1}_R(B,\ C)) \xrightarrow{\beta} \mathrm{Text}^n_R(A,\ B,\ C) \xrightarrow{\alpha} \mathrm{Hom}_R(A,\ \mathrm{Ext}^n_R(B,\ C)) \longrightarrow 0$$

from (2) as the first half of the theorem.

In this case the homomorphisms α and β are decomposed as follows, respectively (see page 81 of [2]).

$$\alpha : \mathrm{Text}^n_R(A,\ B,\ C) \xrightarrow{i^*} H^n(\mathrm{Hom}_R(\mathrm{Ker}(X),\ \mathrm{Hom}_R(Y,\ I))) \xrightarrow{\alpha_n} \mathrm{Hom}_R(X_0,\ H^n(\mathrm{Hom}_R(Y,\ I))$$
$$\cong \mathrm{Hom}_R(X_0,\ \mathrm{Ext}^n_R(B,\ C)) \longrightarrow \mathrm{Hom}_R(A,\ \mathrm{Ext}^n_R(B,\ C)), \qquad (5)$$

where the last arrow stands for the additive relation which is the inverse of the first monomorphism in (4).

$$\beta : \mathrm{Ext}^1_R(A,\ \mathrm{Ext}^{n-1}_R(B,\ C)) \xrightarrow{S^{*-1}} \mathrm{Hom}_R(X_1,\ \mathrm{Ext}^{n-1}_R(B,\ C)) \cong \mathrm{Hom}_R(X_1,\ \mathrm{H}^{n-1}(\mathrm{Hom}_R(Y,\ I)))$$
$$\xrightarrow{\alpha_n^{-1}} H^n(\mathrm{Hom}_R(\mathrm{Coim}(X),\ \mathrm{Hom}_R(Y,\ I))) \xrightarrow{j^*} \mathrm{Text}^n_R(A,\ B,\ C).$$

To show the second half we consider the diagrams (i) and (ii)

(i)

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}(X) & \longrightarrow & \mathrm{Ker}(X) & \longrightarrow & H(X) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}(X) & \longrightarrow & X & \longrightarrow & \mathrm{Cok}(X) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathrm{Coim}(X) & = & \mathrm{Coim}(X) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \\ & & & & \text{(split)} & & \text{(split)} & & \end{array}$$

in (i) and (ii) (below), each column in the first (i) and each row in the second (ii) is split exact and the other rows and columns are exact since Coim(X) is projective and Coim(Hom$_R(Y, I)$) $\cong$ Im(Hom$_R(Y, I)$) injective by the assumption.

(ii)

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}(\mathrm{Hom}_R(Y, I)) & \longrightarrow & \mathrm{Ker}(\mathrm{Hom}_R(Y, I)) & \longrightarrow & H(\mathrm{Hom}_R(Y, I)) & \longrightarrow & 0 \ \text{(split)} \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}(\mathrm{Hom}_R(Y, I)) & \longrightarrow & \mathrm{Hom}_R(Y, I) & \longrightarrow & \mathrm{Cok}(\mathrm{Hom}_R(Y, I)) & \longrightarrow & 0 \ \text{(split)} \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathrm{Coim}(\mathrm{Hom}_R(Y, I)) & = & \mathrm{Coim}(\mathrm{Hom}_R(Y, I)) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

In this situation we get the following commutative diagrams successively.

i)

$$\begin{array}{cccccc} 0 & & 0 & & & \\ \downarrow & & \downarrow & & & \\ \mathrm{Ker}(X) & \longrightarrow & H(X) & \longrightarrow & 0 \\ \downarrow\uparrow & & \downarrow\uparrow\varphi & & \\ X & \longrightarrow & \mathrm{Cok}(X) & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ \mathrm{Ker}(\mathrm{Hom}_R(X, I)) & \overset{\varphi_2}{\rightleftarrows} & H(\mathrm{Hom}_R(Y, I)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}_R(X, I) & \rightleftarrows & \mathrm{Cok}(\mathrm{Hom}_R(Y, I)) & \longrightarrow & 0 \end{array}$$

ii) From i)

$$\begin{array}{ccc} \mathrm{Hom}_R(\mathrm{Cok}(X), \mathrm{Ker}(\mathrm{Hom}_R(Y, I))) & \underset{\mathrm{Hom}(\varphi_1, \varphi_2)}{\overset{\xi\,(\text{epi.})}{\rightleftarrows}} & \mathrm{Hom}_R(\mathrm{H}(X), H(\mathrm{Hom}_R(Y, I))) \\ \eta' \downarrow \text{(mon.)} & \nearrow \alpha & \tau \,|\, \text{(mon.)} \\ \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I)) & \overset{\zeta}{\longrightarrow} & \mathrm{Hom}_R(\mathrm{Ker}(X), \mathrm{Cok}(\mathrm{Hom}_R(Y, I))) \end{array}$$

iii) Taking homology in ii),

$$\begin{array}{ccc} \mathrm{Hom}_R(\mathrm{Cok}(X), \mathrm{Ker}(\mathrm{Hom}_R(Y, I))) & \underset{\mathrm{Hom}(\varphi_1, \varphi_2)}{\overset{\xi\,(\text{epi.})}{\rightleftarrows}} & \mathrm{Hom}_R(H(X), H(\mathrm{Hom}_R(Y, I))) \\ \eta \downarrow & \nearrow \alpha & \tau \,|\, \text{(mon.)} \\ \mathrm{Text}_R(A, B, C) & \overset{\zeta}{\longrightarrow} & \mathrm{Hom}_R(\mathrm{Ker}(X), \mathrm{Cok}(\mathrm{Hom}_R(Y, I))) \end{array}$$

iv)

$$
\begin{array}{ccc}
\mathrm{Hom}_R(\mathrm{Cok}(X),\ \mathrm{Ker}(\mathrm{Hom}_R(Y,\ I))) & \underset{\mathrm{Hom}(\varphi_1,\ \varphi_2)}{\overset{\xi}{\rightleftarrows}} & \mathrm{Hom}_R(H(X),\ H(\mathrm{Hom}_R(Y,\ I))) \\
\eta\downarrow & \alpha & \tau\downarrow(\mathrm{mon.})\ \uparrow\overline{ad} \\
\mathrm{Text}_R(A,\ B,\ C) & \xrightarrow{\ \zeta\ } & \mathrm{Hom}_R(\mathrm{Ker}(X),\ \mathrm{Cok}(\mathrm{Hom}_R(Y,\ I))) \\
i^*\downarrow & & l^*\uparrow(\mathrm{mon.}) \\
H(\mathrm{Hom}_R(\mathrm{Ker}(X),\ \mathrm{Hom}_R(Y,\ I))) & \xrightarrow{\ \cong\ } & \mathrm{Hom}_R(\mathrm{Ker}(X),\ H(\mathrm{Hom}_R(Y,\ I)))
\end{array}
$$

(Note:

$H(X)\oplus\mathrm{Coim}(X)\cong\mathrm{Cok}(X)$ $\mathrm{Im}(\mathrm{Hom}_R(Y,\ I))\oplus H(\mathrm{Hom}_R(Y,\ I))\cong\mathrm{Ker}(\mathrm{Hom}_R(Y,\ I))$

$\mathrm{Ker}(X)\oplus\mathrm{Coim}(X),\ \cong X$ $\mathrm{Im}(\mathrm{Hom}_R(Y,\ I))\oplus\mathrm{Cok}(\mathrm{Hom}_R(Y,\ I))\cong\mathrm{Hom}_R(Y,\ I)$.)

v) In each degree n,

$$
\begin{array}{ccc}
\mathrm{Hom}^n(\mathrm{Cok}(X),\ \mathrm{Ker}(\mathrm{Hom}_R(Y,\ I))) & \underset{\mathrm{Hom}(\varphi_1,\ \varphi_2)}{\overset{\xi}{\rightleftarrows}} & \mathrm{Hom}_R(H^0(X),\ H^n(\mathrm{Hom}_R(Y,\ I))) \\
\eta\downarrow & \alpha & \tau\downarrow(\mathrm{mon.})\ \uparrow\overline{ad} \\
\mathrm{Text}^n_R(A,\ B,\ C) & \xrightarrow{\ \zeta\ } & \mathrm{Hom}_R(X_0,\ \mathrm{Cok}(\mathrm{Hom}_R(Y,\ I))_n) \\
i^*\downarrow & & l^*\uparrow \\
H^n(\mathrm{Hom}_R(X_0,\ \mathrm{Hom}_R(Y,\ I))) & \underset{\alpha_n}{\xrightarrow{\ \cong\ }} & \mathrm{Hom}_R(X_0,\ H^n(\mathrm{Hom}_R(Y,\ I))),
\end{array}
$$

where $\zeta=l^*\alpha_n i^*$, l^* and τ are monomorphisms and $\overline{ad}$ stands for the additive relation which is in the composite of α (the converse of τ). Since $\mathrm{Im}\,\tau\subset\mathrm{Im}\,l^*$ the homomorphism α in the above diagram is the composite $\overline{ad}\cdot\zeta=\overline{ad}\cdot l^*\ \alpha_n\cdot i^*$ and the same one as α in (5).

By the splitting homomorphisms φ_1 and φ_2 we have the right inverse $\eta\cdot\mathrm{Hom}(\varphi_1,\ \varphi_2)$ of α which implies that the exact sequence in our theorem splits. Since $\mathrm{Hom}(\varphi_1,\ \varphi_2)$ has no naturality the isomorphism

$$\mathrm{Text}^n_R(A,\ B,\ C)\cong\mathrm{Ext}^1_R(A,\ \mathrm{Ext}^{n-1}_R(B,\ C))\oplus\mathrm{Hom}_R(A,\ \mathrm{Ext}^n_R(B,\ C))$$

is non-natural. (Note: When $Y\longrightarrow B$ is $0\longrightarrow Y_1\longrightarrow Y_0\longrightarrow B\longrightarrow 0$ (exact) the above exact sequence (in the theorem) becomes

$$0\longrightarrow\mathrm{Ext}^1_R(B,\ \mathrm{Ext}^{n-1}_R(A,\ C))\longrightarrow\mathrm{Text}^n_R(A,\ B,\ C)$$
$$\cong\mathrm{Text}^n_R(B,\ A,\ C)\longrightarrow\mathrm{Hom}_R(B,\ \mathrm{Ext}^n_R(A,\ C))\longrightarrow 0.$$

Moreover, if each quotient of all modules in $\mathrm{Hom}_R(X,\ I)$ is injective the above exact sequence is split(non-natural).)

Corollary 1. *If A (or B) is projective as a R-module then*

$$\mathrm{Text}_R^n(A, B, C) \cong \mathrm{Hom}_R(A, \mathrm{Ext}_R^n(B, C)) \ (\cong \mathrm{Hom}_R(B, \mathrm{Ext}_R^n(A, C)).$$

Proof. Since A is projective we can take $0 \longrightarrow 0 \longrightarrow A \longrightarrow A \longrightarrow 0$ as a projective resolution over A. This implies that $X_1 = 0$, $X_0 = A$ in the above theorem. Therefore

$$0 \longrightarrow \mathrm{Ext}_R^1(A, \mathrm{Ext}_R^{n-1}(B, C)) \longrightarrow \mathrm{Text}_R^n(A, B, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Ext}_R^n(B, C)) \longrightarrow 0$$

is exact. We have therefore $\mathrm{Text}_R^n(A, B, C) \cong \mathrm{Hom}_R(A, \mathrm{Ext}_R^n(B, C)$ since $\mathrm{Ext}_R^1(A, \mathrm{Ext}_R^{n-1}(B, C)) = 0$. When B is projective we can apply the same argument as above.

COROLLARY 2. *Let $X \longrightarrow A$ be $0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$ as in Theorem* 1. *If the projective dimension of B is $\leq n$ (positive integer) then*

$$\mathrm{Text}_R^{n+1}(A, B, C) \cong \mathrm{Ext}_R^1(A, \mathrm{Ext}_R^n(B, C)).$$

Proof. By Theorem 1, the sequence

$$0 \longrightarrow \mathrm{Ext}_R^1(A, \mathrm{Ext}_R^n(B, C)) \longrightarrow \mathrm{Text}_R^{n+1}(A, B, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Ext}_R^{n+1}(B, C)) \longrightarrow 0$$

is exact. Since $\mathrm{Ext}_R^{n+1}(B, C) = 0$ we get

$$\mathrm{Text}_R^{n+1}(A, B, C) \cong \mathrm{Ext}_R^1(A, \mathrm{Ext}_R^n(B, C)).$$

as asserted.

EXAMPLE 2. Let F be a field and let x be an indeterminate. Then we get the polynomial ring $P = F[x]$ which is commutative. We can put $F = F[x]/(x)$, where (x) is the principal ideal consisting of all multiples of x. Therefore F becomes P-module by the P-module homomorphism $\varepsilon: P \longrightarrow F$ which is defined by $\varepsilon(x) = 0$ and $\varepsilon(a) = a$ for $a \in F$. In this case we have the following sequence as a projective resolution over F.

$$0 \longrightarrow PU \xrightarrow{\partial} P \xrightarrow{\varepsilon} F \longrightarrow 0,$$

where PU is the free P-module generated by U and ∂ is the P-module homomorphism with $\partial U = x$. Therefore Theorem 1 is valid in the case which we take F, B, and C as P-modules and the sequence

$$0 \longrightarrow \mathrm{Ext}_P^1(F, \mathrm{Ext}_P^{n-1}(B, C)) \longrightarrow \mathrm{Text}_P^n(F, B, C) \longrightarrow \mathrm{Hom}_P(F, \mathrm{Ext}_P^n(B, C)) \longrightarrow 0$$

is exact. The case which the commutative ring R above is a hereditary ring is an example for the second half of our Theorem 1. We can see this example in the next section.

3. Text over the ring Z of integers

Let A, B and C be abelian groups. We shall take

$$0 \longrightarrow X_1 \longrightarrow X_0 \xrightarrow{\partial_A} A \longrightarrow 0 \qquad \text{(as a projective resolution over } A)$$

$$0 \longrightarrow Y_1 \xrightarrow{\partial_B} Y_0 \longrightarrow B \longrightarrow 0 \qquad \text{(as a projective resolution over } B\text{)},$$

$$0 \longrightarrow C \longrightarrow I^0 \xrightarrow{\partial_C} I^1 \longrightarrow 0 \qquad \text{(as an injective resolution over } C\text{)},$$

then X and Y are free complexes and we get complexes

$$\mathrm{Hom}(X, \mathrm{Hom}(Y, I)) \cong \mathrm{Hom}(X \otimes Y, I)$$

with boundaries ∂_H and $\bar{\partial}_H$ (see § 1), respectively, where Hom and $\otimes$ mean Hom_Z and $\otimes_Z$ (in this section the subscripts Z are omitted). We should note that $\mathrm{Hom}(Y, I)$ is an injective complex and $X \otimes Y$ is a free complex. Moreover, since Z is a hereditary ring each quotient in $\mathrm{Hom}(Y, I)$ and each submodule in $X \otimes Y$ are injective and free, respectively.

LEMMA 3. *With the above situation the following hold.*

(i) $\mathrm{Text}^1(A, B, C) \cong \mathrm{Ext}^1(A, \mathrm{Hom}(B, C)) \oplus \mathrm{Hom}(A, \mathrm{Ext}^1(B, C))$
$\cong \mathrm{Ext}^1(A \otimes B, C) \oplus \mathrm{Hom}(\mathrm{Tor}_1(A, B), C)$ (non-natural)

(ii) $\mathrm{Text}^2(A, B, C) \cong \mathrm{Ext}^1(A, \mathrm{Ext}^1(B, C)) \cong \mathrm{Ext}^1(\mathrm{Tor}_1(A, B), C)$ (natural)

(iii) $\mathrm{Text}^n(A, B, C) = 0$ for $n \geq 3$.

Proof. Since $\mathrm{Hom}^n(X, \mathrm{Hom}(Y, I)) = 0$ for $n \geq 3$ (see § 1) (iii) is true. By the above description we know that $\mathrm{Hom}(X, \mathrm{Hom}(Y, I))$ satisfies the hypothesis of Theorem 1 in § 2 and $\mathrm{Hom}(X \otimes Y, I)$ satisfies the hypothesis of *Homotopy Classification Theorem* (see Theorem 4.3 on page 78 of [2]). Therefore we have two split (non-natural) exact sequences

$$0 \longrightarrow \mathrm{Ext}^1(A, \mathrm{Ext}^{n-1}(B, C)) \longrightarrow \mathrm{Text}^n(A, B, C) \longrightarrow \mathrm{Hom}(A, \mathrm{Ext}^n(B, C)) \longrightarrow 0,$$

$$0 \longrightarrow \prod_{p=-\infty}^{\infty} \mathrm{Ext}^1(H_p(X \otimes Y, H^{n-p-1}(I)) \longrightarrow \mathrm{Text}^n(A, B, C) \longrightarrow$$

$$\prod_{p=-\infty}^{\infty} \mathrm{Hom}(\mathrm{H}_p(X \otimes Y), H^{n-p}(I)) \longrightarrow 0.$$

When we note that $H^n(I) = 0$ for $n \neq 0$ we can easily deduce (ii) and (i) form the above two sequences.

EXAMLE 3. Let $Z_{rs}(a_0)$ be a cyclic group of order rs generated by a_0. Put $A = Z_{rs}(a_0)$, $B = Z_r(b_0)$ and let C be any abelian group. Since $\mathrm{Hom}(Z_m(g_0), G) \cong 0_m(G) = \{g \mid g \in G,\ mg = 0\}$ and $\mathrm{Ext}^1(Z_m(g_0), G) = G/mG$ ($mG = \{mg \mid g \in G\}$) for an abelian group G we know the following using Lemma 3 above.

$$\mathrm{Text}^0(A, B, C) \cong \mathrm{Hom}(Z_{rs}(a_0), \mathrm{Hom}(Z_r(b_0), C)) \cong 0_r(C),$$

$$\mathrm{Text}^1(A, B, C) \cong \mathrm{Ext}^1(Z_{rs}(a_0), \mathrm{Hom}(Z_r(b_0), C)) \oplus \mathrm{Hom}(Z_{rs}(a_0), \mathrm{Ext}^1(Z_r(b_0), C))$$
$$\cong 0_r(C) \oplus C/rC$$

$$\mathrm{Text}^2(A,B,C)\cong \mathrm{Ext}^1(Z_{rs}(a_0),\ \mathrm{Ext}^1(Z_r(b_0),C))\cong C/rC$$

Let K and L be complexes of abelian groups with each K_n and L_n free over the ring Z of integers and let M be a complex of abelian groups with each M_n is injective. From Lemma 3 we have that

$$\mathrm{Text}^0(H_p(K),\ H_q(L), H_{n+p+q}(M))\cong \mathrm{Hom}(H_p(K), \mathrm{Hom}(H_q(L), H_{n+p+q}(M))) \quad \text{(natural)}$$
$$\mathrm{Text}^1(H_p(K), H_q(L), H_{n+p+q}(M))\cong \mathrm{Ext}^1(H_p(K), \mathrm{Hom}(H_q(L), H_{n+p+q}(M))) \oplus \mathrm{Hom}(H_p(K), \mathrm{Ext}^1(H_q(L), H_{n+p+q}(M))) \quad \text{(non-natural)}$$
$$\mathrm{Text}^2(H_p(K),\ H_q(L),\ H_{n+p+q}(M))\cong \mathrm{Ext}^1(H_p(K), \mathrm{Ext}^1(H_q(L), H_{n+p+q}(M))) \quad \text{(natural)}$$
$$\mathrm{Text}^m(H_p(K), H_q(L), H_{n+p+q}(M)) = 0 \quad (\text{for } m\geq 3)$$

for each p, q and n, where $\mathrm{Hom}_n(K,\ \mathrm{Hom}(L, M)) = \prod_{p=-\infty}^{\infty}\cdot\prod_{q=-\infty}^{\infty} \mathrm{Hom}(K_p, \mathrm{Hom}(L_q, M_{n+p+q}))$.

Define

$$\mathrm{Text}^m_n(N(K),\ H(L), H(M)) = \prod_{p=-\infty}^{\infty}\cdot\prod_{q=-\infty}^{\infty} \mathrm{Text}^m(H_p(K), H_q(L),\ H_{n+p+q}(M)).$$

for $m=0$, 1, 2 then the following hold.

THEOREM 2. *Let $S_n=H_n(\mathrm{Hom}(K, \mathrm{Hom}(L, M)))$. Then there are subgroups $0<N_{n+2}<R_{n+1}<S_n$ and isomorphisms*

$$\alpha_{n+2}:\ \mathrm{Text}^2_{n+2}(H(K), H(L), H(M))\cong N_{n+2} \quad (\textit{natural})$$
$$\alpha_{n+1}:\ \mathrm{Text}^1_{n+1}(H(K), H(L), H(M))\cong R_{n+1}/N_{n+2} \quad (\textit{non-natural})$$
$$\alpha_n\ :\ \mathrm{Text}^0_n(H(K), H(L), H(M))\cong S_n/R_{n+1} \quad (\textit{natural})$$

(Note: see §1 for the boundary in $\mathrm{Hom}(K, \mathrm{Hom}(L, M))$.)

Proof. Since K is a projective complex and $\mathrm{Hom}(L, M)$ an injective complex we have the split (non-natural) exact sequences

$$0\longrightarrow \prod_{p=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), H_{n+p+1}(\mathrm{Hom}(L, M)))\longrightarrow H_n(\mathrm{Hom}(K, \mathrm{Hom}(L, M)))$$
$$\longrightarrow \prod_{p=-\infty}^{\infty} \mathrm{Hom}(H_p(K), H_{n+p}(\mathrm{Hom}(L, M)))\longrightarrow 0$$
$$0\longrightarrow \prod_{p=-\infty}^{\infty} \mathrm{Ext}^1(H_q(L), H_{n+p+q+1}(M)) \longrightarrow H_{n+p}(\mathrm{Hom}(L, M))$$
$$\longrightarrow \prod_{p=-\infty}^{\infty} \mathrm{Hom}(H_q(L), H_{n+q+n}(M)) \longrightarrow 0$$

by the *Homotopy Classification Theorem,* where we should know that L is a projective complex and M an injective complex. According to the above two sequences we can make the following diagram.

$$
\begin{array}{ccc}
 & 0 & 0 \\
 & \downarrow & \downarrow \\
\prod_{p=-\infty}^{\infty}\cdot\prod_{q=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), & \mathrm{Ext}^1(H_q(L), H_{n+p+q+2}(M))) & \mathrm{Hom}(H_p(K), \mathrm{Ext}^1(H_q(L), H_{n+p+q+1}(M))) \\
0 \longrightarrow & \downarrow \searrow i & \downarrow \\
\prod_{p=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), H_{n+p+1}(\mathrm{Hom}(L,M))) \to & H_n(\mathrm{Hom}(K, \mathrm{Hom}(L,M))) \to & \mathrm{Hom}(H_p(K) H_{n-p}(\mathrm{Hom}(L,M))) \\
 & \downarrow \searrow j & \downarrow \quad \longrightarrow 0 \text{ (exact)} \\
\prod_{p=-\infty}^{\infty}\cdot\prod_{q=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), & \mathrm{Hom}(H_q(L), H_{n+p+q+1}(M))) & \mathrm{Hom}(H_p(K), \mathrm{Hom}(H_q(L), H_{n+p+q}(M))) \\
 & \downarrow & \downarrow \\
 & 0 \text{ (exact)} & 0 \text{ (exact)}
\end{array}
$$

Therefore,

$$j\colon H_n(\mathrm{Hom}(K, \mathrm{Hom}(I, M)) \longrightarrow \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Hom}(H_p(K), \mathrm{Hom}(H_q(L), H_{n+p+q}(M)))$$

is an epimorphism and

$$i\colon \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), \mathrm{Ext}^1(H_q(L), H_{n+p+q+2}(M))) \longrightarrow H_n(\mathrm{Hom}(K, \mathrm{Hom}(L, M)))$$

is a monomorphism.

Set Ker $j = R_{n+1}$ and Im $i = N_{n+2}$ then

$$R_{n+1} \cong \prod_{p=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), H_{n+p+1}(\mathrm{Hom}(L, M))$$
$$\oplus \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Hom}(H_p(K), \mathrm{Ext}^1(H_q(L), H_{n+p+q+1}(M))) \qquad (1)$$

and

$$N_{n+2} \cong \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), \mathrm{Ext}^1(H_q(L), H_{n+p+q+2}(M))).$$

When we note $\mathrm{Text}_n^0(H(K), H(L), H(M)) \cong \mathrm{Hom}_n(H(K), \mathrm{Hom}(H(L), H(M)))$ (see the definition above) we see that there are natural isomorphisms

$$\alpha_n\colon \mathrm{Text}_n^0(H(K), H(L), H(M)) \cong S/R_{n+1}$$
$$\alpha_{n+2}\colon \mathrm{Text}_{n+2}^2(H(K), H(L), H(M)) \cong N_{n+2}$$

(the naturality of α_n and α_{n+2} is from the naturality of i and j).

From the first column in the above diagram we get

$$\prod_{p=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), H_{p+q+1}(\mathrm{Hom}(L, M)))/ \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), \mathrm{Ext}^1(H_q(L), H_{n+p+q}(M)))$$
$$\cong \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), \mathrm{Hom}(H_q(L), H_{n+p+q+1}(M))). \qquad (2)$$

Combining (2) and (1)

$$R_{n+1}/N_{n+2} \cong \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Ext}^1(H_p(K), \mathrm{Hom}(H_q(L), H_{n+p+q+1}(M)))$$
$$\oplus \prod_{p=-\infty}^{\infty} \prod_{q=-\infty}^{\infty} \mathrm{Hom}(H_p(K), \mathrm{Ext}^1(H_q(L), H_{n+p+q+1}(M))) \text{ (natural)}$$

$$\cong \mathrm{Text}^1_{n+1}(H(K),\ H(L),\ H(M))\ \ (\text{non-natural})$$

as our assertion.

Define

$$\mathrm{Text}^0(H(K),\ H(L),\ H(M)) = \sum_n \mathrm{Text}^0_n(H(K),\ H(L),\ H(M))$$
$$\mathrm{Text}^1(H(K),\ H(L),\ H(M)) = \sum_n \mathrm{Text}^1_n(H(K),\ H(L),\ H(M))$$
$$\mathrm{Text}^2(H(K),\ H(L),\ H(M)) = \sum_n \mathrm{Text}^2_n(H(K),\ H(L),\ H(M))$$

which are direct sums over n, then the following holds.

Corollary 3. *There exist subgroups* $0 < N < R < S$ *and isomorphisms*

$$\alpha_2\colon \mathrm{Text}^2(H(K),\ H(L),\ H(M)) \cong N \qquad (\textit{natural})$$
$$\alpha_1\colon \mathrm{Text}^1(H(K),\ H(L),\ H(M)) \cong R/N \qquad (\textit{non-natural})$$
$$\alpha_0\colon \mathrm{Text}^0(H(K),\ H(L),\ H(M)) \cong S/R \qquad (\textit{natural})$$

where $S = \sum_n S_n$, $R = \sum_n R_n$ and $N = \sum_n N_n$ *(direct sum).*

Proof. It suffices to prove $R_{n+1}/N_{n+2} \oplus R_{n+2}/N_{n+3} \cong (R_{n+1} \oplus R_{n+2})/(N_{n+2} \oplus N_{n+3})$ for some n by Theorem 2. We have the exact sequences

$$0 \longrightarrow N_{n+2} \longrightarrow R_{n+1} \longrightarrow T_1 \longrightarrow 0, \quad 0 \longrightarrow N_{n+3} \longrightarrow R_{n+2} \longrightarrow T_2 \longrightarrow 0$$

where $T_1 \cong R_{n+1}/N_{n+2}$ and $T_2 \cong R_{n+2}/N_{n+3}$. Since

$$0 \longrightarrow N_{n+2} \oplus N_{n+3} \longrightarrow R_{n+1} \oplus R_{n+2} \longrightarrow T_1 \oplus T_2 \longrightarrow 0$$

is exact we proved

$$(R_{n+1} \oplus R_{n+2})/(N_{n+2} \oplus N_{n+3}) \cong T_1 \oplus T_2 \cong R_{n+1}/N_{n+1} \oplus R_{n+2}/N_{n+3}$$

as required.

4. Applications of Spectral Sequences to Text

Let R be a commutative ring and K a complex of R-modules with the boundary ∂_K and filtration F such that for an integer p

$$\cdots\cdots\cdots \supset F^pK \supset F^{p+1}K \supset \cdots\cdots\cdots,\ \ \partial_K(F^pK) \subset F^pK.$$

In this case there is a spectral sequence $\{E_r,\ d_r\}$, $r = 1, 2, \cdots\cdots\cdots$ which is a covariant functor of $(F,\ K)$, together with natural isomorphisms

$$E^p_1 \cong H(F^pK/F^{p+1}K),\ \textit{i.e.},\ E^{p,q}_1 = H^{p+q}(F^pK/F^{p+1}K).$$

In particular, if F is bounded (or convergent below and bounded above) $\{E_r, d_r\}$ converges to $H(K)$, *i.e.*, $E^p_2 \Longrightarrow H(K)$ (see page 327 of [2]). More explicity,

$$E_\infty^p \cong F^p(H(K))/F^{p+1}(H(K)), \ i.e., \ E_\infty^{p,q} \cong F^p(H^{p+q}(K))/F^{p+1}(H^{p+q}(K)),$$

where $F^p(H(K))$ means the image of the map $H(F^pK) \longrightarrow H(K)$ induced by the injection $F^pK \longrightarrow K$.

In detail: Define $\bar{Z}_r^p = \{a \in F^pK \mid \partial_K a \in F^{p+r}K\}$ and the canonical projection η_p: $F^pK \longrightarrow F^pK/F^{p+1}K$. Then

$$E_r^p = \eta_p \bar{Z}_r^p / \eta_p(\partial_K \bar{Z}_{r-1}^{p-r}), \ i.e., \ E_r^{p-q} = \eta_p \bar{Z}_r^{p,q} / \eta_p(\partial_K \bar{Z}_{r-1}^{p-r,q+r-1})$$

(see page 328 of [2]). Put

$$\bar{C}_r^{p,q} = \eta_p \bar{Z}_r^{p,q}, \quad \bar{B}_r^{p,q} = \eta_p(\partial_K \bar{Z}_{r-1}^{p-r,q+r-1})$$

then $E_r^{p,q} = \bar{C}_r^{p,q}/\bar{B}_r^{p,q}$ or $E_r^p = \bar{C}_r^p/\bar{B}_r^p$ and there is a tower

$$\bar{B}_0^p \subset \bar{B}_1^p \subset \cdots\cdots \subset \bar{B}_r^p \subset \cdots\cdots \subset \bar{C}_r^p \subset \cdots\cdots \subset \bar{C}_1^p \subset \bar{C}_0^p = E_0^p,$$

where $\bar{B}_0^p = 0$, $\bar{C}_0^p = E_0^p = F^pK/F^{p+1}K$. In this case $d_r : E_r^p \longrightarrow E_r^{p+r}$ is defined by the composite

$$E_r^p = \bar{C}_r^p/\bar{B}_r^p \xrightarrow{\text{projection}} \bar{C}_r^p/\bar{C}_{r+1}^p \cong \bar{B}_{r+1}^{p+r}/\bar{B}_r^{p+r} \xrightarrow{\text{injection}} \bar{C}_r^{p+r}/\bar{B}_r^{p+r} = E_r^{p+r}$$

hence $\operatorname{Ker} d_r^p = \bar{C}_{r+1}^p/\bar{B}_r^p$ and $\operatorname{Im} d_r^p \cong \bar{B}_{r+1}^{p+r}/\bar{B}_r^{p+r}$, *i.e.*,

$$\operatorname{Ker} d_r^p \cong \bar{C}_{r+1}^{p,q}/\bar{B}_{r+1}^{p,q}, \qquad \operatorname{Im} d_r^{p,q} = \bar{B}_r^{p+r,q-r+1}/\bar{B}_r^{p+r,q-r+1} \tag{1}$$

(see page 329 of [2]).

LEMMA 3. *If* $E_r^{p-s,q+s-1} = 0$ *for* $r = s < \infty$ *then the sequence*

$$0 \longrightarrow E_{s+1}^{p,q} \longrightarrow E_s^{p,q} \xrightarrow{d_s^{p,q}} E_s^{p+s,q-s+1}$$

is exact.

Proof. Put $r = s$ then $E_s^{p-s,q+s-1} = 0$ by our assumption, which means $\operatorname{Im} d_{s+1}^{p-s,q+s-1} \cong \bar{B}_s^{p,q}/\bar{B}_s^{p,q} = 0$ (see (1)). Hence

$$0 \longrightarrow E_{s+p}^{p,q}(\cong \bar{C}_{s+1}^{p,q}/\bar{B}_{s+1}^{p,q}) \xrightarrow{i} E_s^{p,q}(\cong \bar{C}_s^{p,q}/\bar{B}_s^{p,q})$$

is a monomorphism and $i(E_{s+1}^{p,q}) \cong \bar{C}_{s+1}^{p,q}/\bar{B}_s^{p,q}$ which is isomorphic to $\operatorname{Ker} d_r^{p,q}$(see (1)). Therefore the following sequence is exact

$$0 \longrightarrow E_{s+1}^{p,q} \xrightarrow{i} E_s^{p,q} \xrightarrow{d_s^{p,q}} E_s^{p+s,q-s+1}.$$

As before, let $\operatorname{Hom}_R(X, \operatorname{Hom}_R(Y, I))$ be a complex which is constructed from a projective resolution X over R-module A, a projective resolution Y over R-module B and injective resolution over R-module C, where we asssume that $\partial_H, \partial_H', \partial_A$ are the boundaries in $\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I))$, $\mathrm{Hom}_R(Y, I)$ and X, respectively. Set

$$K = \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I)), \quad T^{p,q} = \mathrm{Hom}_R(X_p, \mathrm{Hom}^q(Y, I)),$$
$$K^n = \sum_{p+q=n} T^{p,q},$$

where $\mathrm{Hom}^q(Y, I) = \sum_{m+n=q} \mathrm{Hom}_R(Y_m, I^n)$. Then we can define a filtration F of K by

$$F^pK = \sum_{r \geq p}^{\infty} \sum_{q=0}^{\infty} T^{r,q} \subset K, \ i.e., \ (F^pK)^n = \sum_{r=p}^{n} T^{r,\, n-r} \subset K^n.$$

Let $f = (\cdots\cdots, 0, f_r, \cdots\cdots, f_n, 0, \cdots\cdots)$ be in F^pK and in K^n, where $f_p : X_p \to \mathrm{Hom}^{n-p}(Y, I)$, etc.. Since

$$(\partial_H f)_p(x_p) = \partial_H'(f_p x_p), \quad (\partial_H f)_{p+1}(x_{p+1}) = \partial_H'(t_{p+1} x_{p+1}) + (-1)^{n+1} f_p(\partial_A x_{p+1}),$$
$$\cdots\cdots, \quad (\partial_H f)_{n+1}(x_{n+1}) = (-1)^{n+1} f_n(\partial_A x_{n+1})$$

for $x_p \in X_p$, and so on, we get

$$\partial_H f = (\cdots\cdots, 0, \ \partial_H' f_p, \ \partial_H' f_{p+1} + (-1)^{n+1} f_p \partial_A, \cdots\cdots, (-1)^{n+1} f_n \partial_A, \ 0, \cdots\cdots),$$

where $\partial_H' f_p : X_p \to \mathrm{Hom}^{n-p+1}(Y, I)$, $\partial_H' f_{p+1} + (-1)^{n+1} f_p \partial_A : X_{p+1} \to \mathrm{Hom}^{n-p}(Y, I)$, and so on. But, since $\mathrm{Hom}_R(X_p, \mathrm{Hom}^{n-p+1}(Y, I),)$, $\cdots\cdots$, $\mathrm{Hom}_R(X_{n+1}, \mathrm{Hom}^0(Y, I))$ all are in F^pK we have $\partial_H f \in F^pK$ for every $f \in F^pK$. Therefore F is well defind as a filtration of K and (F, K) determines a spectral sequence such that

$$E_1^p = H(F^pK/F^{p+1}K), \ i.e., \ E_1^{p,q} = H^{p+q}(F^pK/F^{p+1}K).$$

Intuitively, we can see the following properties.

(i) $T^{p,q} = 0$ for $p < 0$ or $q < 0$ and $F^pK = K$ if $p \leq 0$.

(ii) $F^pK/F^{p+1}K = \sum_q T^{p,q} = \sum_q \mathrm{Hom}_R(X_p, \mathrm{Hom}^q(Y, I))$ which is a complex such that

$$\mathrm{Hom}_R(X_p, \mathrm{Hom}^0(Y, I)) \xrightarrow{\partial_H} \mathrm{Hom}_R(X_p, \mathrm{Hom}^1(Y, I)) \xrightarrow{\partial_H} \cdots\cdots$$

(Of course, if $p < 0$ then $F^pK/F^{p+1}K = 0$). For example, we can consider $f_p \in \mathrm{Hom}_R(X_p, \mathrm{Hom}^0(Y, I))$ as a $f = (\cdots\cdots, 0, f_p, 0, \cdots\cdots) \in K^p$ and for $x_p \in X_p$ we get $\partial_H f(\cdots\cdots, 0, x_p, 0, \cdots\cdots) = \partial_H'(f_p x_p)$. This means that the boundary in $F^pK/F^{p+1}K$ is epual to ∂_H' which is the boundary in $\mathrm{Hom}_R(X, Y)$.

On the other hand, since $\sum_{p=0}^{\infty}(F^pK/F^{p+1}K) = K$ (with the boundary ∂_H') we get

$E_1 = \sum_{p=0}^{\infty} E_1^p = H'(K)$, where H' is the homology functor for the boundaty ∂_H'.

In $H'(K)$ the boundary becomes zero we can get $H(E_1) = H''(H'(K)) \cong E_2 = \sum_{p=0}^{\infty} E_2^p$, where H 's the homology for d_1 and H'' the homology for ∂_H which has sign $\pm$

In consequence

$E_2^{p,q} \cong H''^p(H'^q(K))$, *i.e.*, $E_2^{p,q} \cong \mathrm{Ext}_R{}^p(A, \mathrm{Ext}_R{}^q(B, C))$. The detail: H'^q $(\mathrm{Hom}_R(X_q, \mathrm{Hom}_R(Y, I)) \cong \mathrm{Hom}_R(X_p, H'^q(\mathrm{Hom}_R(Y, I)) \cong \mathrm{Hom}_R(X_p, \mathrm{Ext}_R{}^q(B, C))$. (Note: In the case which X_p is projective $\mathrm{Hom}_R(X_p, -)$ is an exact functor and $H(\mathrm{Hom}_R(X_p, Y)) \cong \mathrm{Hom}_R(X_p, H(Y))$ for a complex Y of R-module.) Next $H''^p(H'^q(\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, I))) \cong H''^p(\mathrm{Hom}_R(X, \mathrm{Ext}_R{}^q(B, C))) = \mathrm{Ext}_R{}^q(A, \mathrm{Ext}_R{}^q(B, C))$ because X is projective resolution over A and $\mathrm{Ext}_R{}^q(B, C)$ is a fixed R-module.

(iii) Since $F^0K = K$ and $F^{n+1}K^n = 0$ for each degree n our filtration F is both convergent below and bounded above. Therefore our spectral sequence $\{E_r, d_r\}$ converges to $H(K)$, *i.e.*,

$$\mathrm{Ext}_R{}^p(A, \mathrm{Ext}_R{}^q(B, C)) \Longrightarrow \mathrm{Text}_R^{p+q}(A, B, C),$$

where the filtration degree is p.

(iv) For $p < 0$, $F^pK/F^{p+1}K = 0$ and for $q < 0$, $H^{p+q}(F^pK) = 0$. That is, $H^{p+q}(F^pK)$ is equal to $\mathrm{Ker}\, \partial_H{}^2 / \mathrm{Im}\, \partial_H{}'$ in the sequence

$$K^{p+q-1} \cap F^pK \xrightarrow{\partial_H'} K^{p+q} \cap F^pK \xrightarrow{\partial_H^2} K^{p+q+1} \cap F^pK.$$

$H^{p-q}(F^pK) = 0$ since for $q < 0$, $K^{p+q} \cap F^pK =$ empty, where $\partial_H{}'$ and $\partial_H{}^2$ are from ∂_H.

With the above preparation we shall prove

Theorem 3. *There exists an exact sequence*

$$0 \longrightarrow \mathrm{Ext}_R{}^1(A, \mathrm{Hom}_R(B, C)) \longrightarrow \mathrm{Text}_R{}^1(A, B, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Ext}_R{}^1(B, C))$$
$$\longrightarrow \mathrm{Ext}_R{}^2(A, \mathrm{Hom}_R(B, C)) \longrightarrow \mathrm{Text}_R{}^2(A, B, C)$$

and homomorphisms

$$\mathrm{Ext}_R{}^n(A, \mathrm{Hom}_R(B, C)) \longrightarrow \mathrm{Text}_R{}^n(A, B, C) \longrightarrow \mathrm{Hom}_R(B, \mathrm{Ext}_R{}^n(B, C)).$$

Poorf. By the condition (iv) above our spectral sequence $\{E_r, d_r\}$ is first quadrant and there are then edge homomorphisms

$$E_\infty^{0,q} = E_{q+2}^{0,q} \longrightarrow E_{q+2}^{0,q} \longrightarrow \cdots\cdots \longrightarrow E_2{}^{0,p} \qquad \text{(monomorphisms)} \qquad (2)$$

$$E_2{}^{p,0} \longrightarrow E_3{}^{p,0} \longrightarrow \cdots\cdots \longrightarrow E_p{}^{p,0} \longrightarrow E_{p+1}^{p,0} = E_\infty^{p,0} \quad \text{(epimorphisms)} \qquad (3)$$

Since $E_\infty^{p,q} = F^p(H^{p+q}(K))/F^{p+1}(H^{p+q}(K))$ (see the first part of this section), $F^0(H^n(K)) = H^n(K)$ and $F^{n+1}(H^n(K)) = 0$ we have

$$E_2{}^{n,0} \xrightarrow{\text{epi.}} E_\infty^{n,0}(\cong F^n(H^n(K))) \xrightarrow{\text{mon.}} H^n(K) \qquad \text{(by (3))}$$

$$H^n(K) \xrightarrow{\text{epi.}} E_\infty^{0,n}(\cong H^n(K)/F^1(H^n(K))) \xrightarrow{\text{mon.}} E_2^{0,n} \text{ (by (2))}$$

Putting $E_2^{n,0} \cong \mathrm{Ext}_R^n(A, \mathrm{Hom}_R(B, C))$, $E_2^{0,n} = \mathrm{Hom}_R(A, \mathrm{Ext}_R^n(B, C))$ into the above sequences we then get

$$\mathrm{Ext}_R^n(A, \mathrm{Hom}_R(B, C)) \xrightarrow{\text{epi.}} E_\infty^{n,0} \xrightarrow{\text{mon.}} \mathrm{Text}_R^n(A, B, C) \xrightarrow{\text{epi.}} E_\infty^{n,0} \xrightarrow{\text{mon.}} \mathrm{Hom}_R(\mathrm{Ext}_R^n(B, C))$$

as asserted in the latter half of the theorem.

Take $n=1$ in (2) and $n=1,\ 2$ in (3), then we have

$$E_\infty^{0,1}(\cong H^1(K)/F^1(H^1(K))) \longrightarrow E_2^{0,1} \quad \text{(monomorphism)},$$

$$E_\infty^{1,0} = E_2^{1,2} \cong F^1(H^1(K)), \quad E_\infty^{2,0} = E_3^{2,0} \cong F^2(H^2(K)), \text{ respectively,}$$

hence we have the sequence

$$0 \longrightarrow E_2^{1,0}(\cong F^1(H^1(K))) \longrightarrow H^1(K) \longrightarrow E_3^{0,1}(= H^1(K)/F^1(H^1(K))) \xrightarrow{\text{mon.}} E_2^{0,1},$$

$$E_2^{2,0} \xrightarrow{\text{epi.}} E_3^{2,0}(\cong F^2(H^2(K))) \xrightarrow{\text{mon.}} H^2(K).$$

Therefore our proof requires to prove that two sequences

$$0 \longrightarrow E_3^{0,1} \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0}, \quad E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow E_3^{2,0}(\xrightarrow{\text{mon.}} H^2(K))$$

are exact.

In Lemma 3, put $p=0$, $q=1$ and $r=s=2$ then $E_s^{p-s,\,q+s-1} = E_2^{-2,\,0} = 0$ we therefore get the exact sequence

$$0 \longrightarrow E_3^{0,1} \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0}.$$

On the other hand, since $E_2^{2,0} \xrightarrow{d_2^{2,0}} E_0^{4,\,-1} = 0$ we have

$$\mathrm{Ker}\, d_2^{2,0} \cong \bar{C}_3^{2,0} / \bar{B}_2^{2,0} \cong E_2^{2,0} = \bar{C}_0^{2,0} / \bar{B}_2^{2,0}$$

(see (1)) and $\bar{C}_3^{2,0} = \bar{C}_2^{2,0}$. In the sequence

$$E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{\alpha} E_3^{2,0}$$

since $\mathrm{Im}\ d_2^{0,1} \cong \bar{B}_3^{2,0} / \bar{B}_2^{2,0}$, $E_2^{2,0} \cong \bar{C}_2^{2,0} / \bar{B}_2^{2,0}$ and $E_3^{2,0} \cong \bar{C}_3^{2,0} / \bar{B}_3^{2,0} = \bar{C}_2^{2,0} / \bar{B}_3^{2,0}(\bar{B}_3^{2,0} \supset \bar{B}_2^{2,0})$, we shall define α by the canonical projection $\bar{C}_2^{2,0} / \bar{B}_2^{2,0} \longrightarrow \bar{C}_2^{2,0} / \bar{B}_3^{2,0}$. Then

$$\mathrm{Ker}\ \alpha \cong \bar{B}_3^{2,0} / \bar{B}_2^{2,0}(\cong \mathrm{Im}\ d_2^{0,1})$$

and the sequence

$$E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{\alpha} E_3^{2,0}$$

is exact. In consequnce the sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(K) \xrightarrow{\mu} E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{\tau} H^2(K)$$

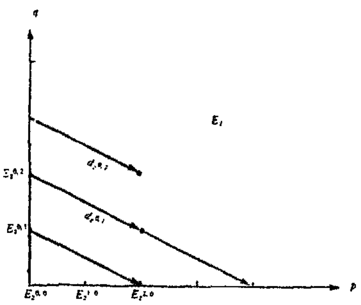
is exact where $\mu: H^1(K) \xrightarrow{\text{proj.}} E_3^{0,1} \xrightarrow{\text{mon.}} E_2^{0,1}$ and $\tau: E_2^{2,0} \longrightarrow E_3^{2,0} \xrightarrow{\text{mon.}} H^2(K)$. Put

$$E_2^{1,2} = \operatorname{Ext}_R^1(A, \operatorname{Hom}_R(B, C)),\ H^1(K) = \operatorname{Text}_R^1(A, B, C),$$

$$E_2^{0,1} = \operatorname{Hom}_R(A, \operatorname{Ext}_R^1(B, C)),\ E_2^{2,0} = \operatorname{Ext}_R^2(A, \operatorname{Hom}_R(B, C))$$

$$\text{and } H^2(K) = \operatorname{Text}_R^2(A, B, C)$$

into the above sequence we then get the exact sequence in the theorem.



The following diagram is helpful for us to understand the above proof, where we can know that

i) $E_2^{0,0} = \operatorname{Hom}_R(A, \operatorname{Hom}_R(B, C))$
$E_2^{1,0} = \operatorname{Ext}_r^1(A, \operatorname{Hom}_R(B, C))$
....

ii) $E_3^{0,1} = \operatorname{Ker} d_2^{0,1},\ E_3^{2,0} = E_2^{2,0}/\operatorname{Im} d_2^{0,1},$
$E_2^{1,0} = E_3^{1,0} = \cdots\cdots = E_\infty^{1,0},\ \cdots\cdots$

Therefore we have the diagram

$$\begin{array}{l}
\downarrow \\
E_\infty^{1,0} = E_2^{1,0} = \operatorname{Ext}_R^1(A, \operatorname{Hom}_R(B, C)) \\
\quad \downarrow \text{mon.} \\
\operatorname{Text}_R^1(A, B, C) \searrow \\
\quad \downarrow \\
E_\infty^{0,1} \longrightarrow E_2^{0,1} = \operatorname{Hom}_R(A, \operatorname{Ext}_R^1(B, C) \xrightarrow{d_2^{0,4}} \\
\quad \downarrow \\
\quad 0
\end{array}$$

$$\begin{array}{rl}
 & 0 \\
 & \downarrow \\
E_2^{2,0} = \operatorname{Ext}_R^2(A, \operatorname{Hom}_R(B, C)) \longrightarrow & E_\infty^{2,0} \longrightarrow 0 \\
\searrow & \downarrow \\
 & \operatorname{Text}_R^2(A, B, C)
\end{array}$$

As a special case, let $X \longrightarrow A$ be $0 \longrightarrow X_1 \xrightarrow{\partial_A} X_0 \longrightarrow A \longrightarrow 0$ (a projective resolution over A). We have then the same exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^1(A, \operatorname{Ext}_R^{n-1}(B, C)) \longrightarrow \operatorname{Text}_R^n(A, B, C) \longrightarrow \operatorname{Hom}_R(A, \operatorname{Ext}_R^n(B, C)) \longrightarrow 0$$

as in Theorem 1 which can be proved using the latter half of the above theorem. Since $E_r^{p,q} = 0$ for $p \neq 0$ or 1 and $r = 1, 2, \cdots\cdots$, we have the exact sequence

$$0 \longrightarrow E_\infty^{1,n-1} (\cong F^1(H^n(K))) \longrightarrow H^n(K) \longrightarrow E_\infty^{0,n} (\cong H^n(K)/F^1(H^n(K))) \longrightarrow 0.$$

By the way, in the sequence

$$E_2^{-1,n} \xrightarrow{d_2^{-1,n}} E_2^{1,n-1} \xrightarrow{d_2^{1,n-1}} E_2^{3,n-2}, \quad E_2^{-2,n+1} \xrightarrow{d_2^{-2,n+1}} E_2^{0,n} \xrightarrow{d_2^{0,n}} E_2^{2,n-1},$$

$d_2^{-1,n} = d_2^{1,n-1} = 0 = d_2^{-2,n+1} = d_2^{0,n}$, hence $E_2^{1,n-1} = E_\infty^{1,n-1}$ and $E_2^{0,n} = E_\infty^{0,n}$. Therefore

$$0 \longrightarrow E_2^{1,n-1} \longrightarrow H^n(K) \longrightarrow E_2^{0,n} \longrightarrow 0, \ \ i.e.,$$

$$0 \longrightarrow \mathrm{Ext}_R^1(A, \ \mathrm{Ext}_R^{n-1}(B, \ C)) \longrightarrow \mathrm{Text}_R^n(A, B, C) \longrightarrow \mathrm{Hom}_R(A, \ \mathrm{Ext}_R^n(B, C) \longrightarrow 0$$

is exact.

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