J. Korean Math. Soc. 6 (1969). 75—80

On Boolean algebraic ideals of Boolean algebras

HYUNG KOO CHA

In Boolean algebras, we know well that it is very hard to define the concept of exact sequences. In this paper, we shall define the concept of Boolean algebraic ideals and by using this we shall introduce the definition of exact sequences of Boolean algebras. We shall try to search properties of exact sequences of Boolean algebras and the coincidence of usual retracts and our retracts.

1. Let B be a set with two distinct distinguished elements 0 and 1, two binary operations $\vee$ and $\wedge$, and a unary operation $'$. If the following conditions (1)~(10) are satisfied, then B is called a *Boolean algebra* [1].

(1)	$0'=1$	$1'=0$
(2)	$p\wedge 0=0$	$p\vee 1=1$
(3)	$p\wedge 1=p$	$p\vee 0=p$
(4)	$p\wedge p'=0$	$p\vee p'=1$
(5)	$p''=p$, where	$p''=(p')'$
(6)	$p\wedge p=p$	$p\vee p=p$
(7)	$(p\wedge q)'=p'\vee q'$	$(p\vee q)'=p'\wedge q'$
(8)	$p\wedge q=q\wedge p$	$p\vee q=q\vee p$
(9)	$p\wedge(q\wedge r)=(p\wedge q)\wedge r$	$p\vee(q\vee r)=(p\vee q)\vee r$
(10)	$p\wedge(q\vee r)=(p\wedge q)\vee(p\wedge r)$	
	$p\vee(q\wedge r)=(p\vee q)\wedge(p\vee r)$,	

where p, q and r are in B.

Let A and B be two Boolean algebras. Suppose that there is a mapping f from A to B, *i.e.*,

$$f: A \longrightarrow B$$

If f preserves the operations $\wedge$, $\vee$ and $'$ in A such that for p and q in A

$$f(p\wedge q)=f(p)\wedge f(q),\quad f(p\vee q)=f(p)\vee f(q),\quad f(p')=(f(p))'$$

then f is called a *Boolean homomorphism from* A to B [1]. We shall denote the set of all Boolean homomorphisms from A to B by $H_\beta(A, B)$.

PROPOSITION 1. *Let f be an element in $H_\beta(A, B)$. Then*

$$f(1)=1 \text{ and } f(0)=0$$

Presented to the Society on October 24, 1969; received by the editor on October 30, 1969.

Proof. For p and q in A

$$f(p\wedge p')=f(0)=f(p)\wedge(f(p))'=0,$$
$$f(p\vee p')=f(1)=f(p)\vee(f(p))'=1.$$

This proposition implies that for each $f\in H_\beta(A,B)$, $f(A)$ is a subalgebra of B.
We shall define the operations $\wedge$ and $\vee$ in $H_\beta(A,B)$ by

$$(f\wedge g)(p)=f(p)\wedge g(p)$$
$$(f\vee g)(p)=f(p)\vee g(p)$$

for $f,\ g\in H_\beta(A,B)$ and for all $p\in A$. We also define the order $\geqq$ in $H_\beta(A,B)$ by

$$f\vee g=f \Longrightarrow f\geqq g$$

for $f,\ g\in H_\beta(A,B)$, where $f\vee g=f$ means that for all p in A, $(f\vee g)(p)=f(p)$. With this order $H_\beta(A,B)$ becomes a partially ordered set. In this case, for each pair f and g in $H_\beta(A,B)$

$$f\vee g = \text{g.l.b. of } f \text{ and } g$$
$$f\wedge g = \text{l.u.b. of } f \text{ and } g \text{ [2]}.$$

Therefore $H_\beta(A,B)$ is a lattice.

PROPOSITION 2. *$H_\beta(A,B)$ is a distributive lattice.*

Proof: For f, g and h in $H_\beta(A,B)$ and all p in A

$$\begin{aligned}[f\vee(g\wedge h)](p)&=f(p)\vee(g\wedge h)(p)\\&=f(p)\vee[g(p)\wedge h(p)]\\&=[f(p)\vee g(p)]\wedge[f(p)\vee h(p)]\\&=(f\vee g)(p)\wedge(f\vee h)(p)\\&=[(f\vee g)\wedge(f\vee h)](p)\end{aligned}$$

and therefore

$$f\vee(g\wedge h)=(f\vee g)\wedge(f\vee h).$$

Similarly we can prove that

$$f\wedge(g\vee h)=(f\wedge g)\vee(f\wedge h).$$

Let $\boldsymbol{B}_A$ be the category consisting of all Boolean algebras and Booean homomorphisms and let $\boldsymbol{D}_L$ be the category consisting of all distributive lattices and lattice homomorphisms (they preserve order and operations $\wedge$, $\vee$). Then

$$\begin{aligned}H_\beta:\ \boldsymbol{B}_A\times\boldsymbol{B}_A &\longrightarrow \boldsymbol{D}_L\\(A,\ B) &\longrightarrow H_\beta(A,\ B)\end{aligned}$$

is a functor which is contravariant in the first argument and covariant in the second argument.

2. Let B be a Boolean algebra. If a subset D of B satisfies the conditions:

$$0 \in D$$
$$p,\ q \in D \Longrightarrow p \vee q \in D$$
$$p \in D,\ q \in B \Longrightarrow p \vee q \in D$$

then D is called an *ideal* of B [1]. Each ideal of B is defined by the kernel of a Boolean epimorphism from B to a suitable Boolean algebra.

THEOREM 1. *Let D be a proper ideal of the Boolean algebra B ($D \subsetneqq B$). Then the quotient B/D becomes a Boolean algebra with suitable operations.*

Proof: We define the operations $\wedge$ and $\vee$ in B/D as follows. Since

$$B/D \ni [p] = p \vee D = \{p \vee d \mid d \in D\}$$

for $[p]$, $[q] \in B/D$, we can define

$$[p] \vee [q] = [p \vee q],$$
$$[p] \wedge [q] = [p \wedge q].$$

These are welldefined because of

$$[p] \vee [q] = (p \vee D) \vee (q \vee D) = (p \vee q) \vee D$$
$$\begin{aligned}[p] \wedge [q] &= (p \vee D) \wedge (q \vee D) \\ &= [(p \vee D) \wedge q] \vee [(p \wedge D) \wedge D] \\ &= [(p \wedge q) \vee (D \wedge q)] \vee D \\ &= (p \wedge q) \vee D \vee D \\ &= (p \wedge q) \vee D.\end{aligned}$$

Since $D \neq B$, 1 is not contained in D. Therefore $[1] \in B/D$ and also $[0] \in B/D$. On the other hand, since

$$(p \vee D)' = p' \wedge D' = \{p' \wedge d' \mid d \in D\}$$

and for all $d \in D$

$$\begin{aligned}p' &= p' \wedge (d \vee d') \\ &= (p' \wedge d) \vee (p' \wedge d') \in D \vee (p' \wedge d'),\end{aligned}$$

we have

$$p' \vee D = \{(p' \wedge D') \vee D \mid d \in D\}.$$

This implies that $[p'] = [p]'$, because, for all $d \in D$

$$[p'] = [p' \wedge d'].$$

We can easily show that our operations $\wedge$, $\vee$, $'$ and distinguished elements $[1]$, $[0]$ satisfy the conditions (1)$\sim$(10) for a Boolean algebra. Hence B/D is a Boolean algebra.

Let M be a subalgebra of the Boolean algebra B. We shall denote the set excepted 1 from M by M^o. If M^o is an ideal of B then we call M a *Boolean algebraic ideal* of B. We know that a Boolean algebraic ideal exists as follows.

EXAMPLE. Let B be a Boolean algebra. Then $I=\{0, 1\}$ is a proper subalgebra of B. In this case, $I^o=\{0\}$. Of course, $I^o=\{0\}$ is an ideal of B. Therefore I is a Boolean algebraic ideal of B.

THEOREM 2. *If M is a Boolean algebraic ideal of the Boolean algebra B and if $p\wedge q=0$ for all $p\in M$ and $q\in M'$ then*

$$B \cong M\oplus B/M^o$$

where $\cong$ means to be an isomorphism in the sense of Boolean algebras and $\oplus$ is a direct sum.

Proof: Since

$$M\oplus B/M^o = \{(p, [q]) \mid p \in M, [q] \in B/M^o\},$$

with term wise-operations $M\oplus B/M^o$ becomes a Boolean algebra [1]. In this Boolean algebra the distinguished elements are $(0, [0])$ and $(1, [1])$. (Note: B/M^o is a Boolean algebra by Theorem 1.)

We define a Boolean homomorphism

$$f\colon B \longrightarrow M\oplus B/M^o$$

by

$$\begin{array}{ll} f(0)=(0, [0]), & f(1)=(1, [1]) \\ f(p)=(p, [0]) & \text{if } p \in M \\ f(q)=(0, [q]) & \text{if } q \in [q] \in B/M^o \\ f(a)=(p, [q]) & \text{if } a \in [q] \text{ and } a=p\vee q \text{ for } p \in M. \end{array}$$

Moreover, for $p \in M$ and $q \in M'=\{a' \mid a\in M\}$,

$$f(p\wedge q)=f(p)\wedge f(q)=(p, [0])\wedge(0, [q])=(0, [0])=f(0)$$

Therefore, we can easily prove that the homomorphism f is an isomorphism.

3. For a Boolean homomorphism

$$f\colon A \longrightarrow B,$$

we define that $f|A^o=f^o$. Let $i\colon A\longrightarrow B$ and $j\colon B\longrightarrow C$ be two Boolean homomorphisms. If the short sequence

$$0 \longrightarrow A^o \xrightarrow{i^o} B^o \xrightarrow{j^o} C^o \longrightarrow 0$$

is exact then we call the sequence $A \xrightarrow{i} B \xrightarrow{j} C$ an *exact sequence* of Boolean algebras A, B and C. We shall denote this by

$$A \overset{i}{\rightarrowtail} B \overset{j}{\twoheadrightarrow} C.$$

THEOREM 3. *In an exact sequence*

$$A \overset{i}{\rightarrowtail} B \overset{j}{\twoheadrightarrow} C,$$

the following hold.

(i) *$i(A)$ is a Boolean algebraic ideal of B.*

(ii) *If, for $p \in i(A)$ and $q \in (i(A))'$, $p \wedge q = 0$ then the exact sequence $A \overset{i}{\rightarrowtail} B \overset{j}{\twoheadrightarrow} C$ is split, where split means $B \cong A \oplus C$.*

PROOF : Under our assumption, the sequence

$$0 \longrightarrow A^o \overset{i^o}{\longrightarrow} B^o \overset{j^o}{\longrightarrow} C^o \longrightarrow 0$$

is exact. Hence $i(A^o)$ is the kernel of the Boolean epimorphism j: $B \longrightarrow C$, therefore $i(A^o)$ is an ideal of B. Hence $i(A)$ is a Boolean algebraic ideal of B.

By our assumption (ii) and Theorem 2 we have the commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A^o \arrow[r, "i^o"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\text{commutes}"] & B^o \arrow[r, dashed, "j^o"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\text{commutes}"] & C^o \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A^o \arrow[r] & A^o \oplus B^o/A^o \arrow[r] & B^o/A^o \arrow[r] & 0
\end{tikzcd}$$

Therefore, by the five-lemma [3] we have

$$C^o \cong B^o/A^o$$

and therefore $A \oplus C \cong B$. This means that the exact sequence

$$A \overset{i}{\rightarrowtail} B \overset{j}{\twoheadrightarrow} C$$

is split.

By using the concept of Boolean algebraic ideals we can easily define the concepts of retracts, projective and injective Boolean algebras [1].

EXAMPLE. Let f: $B \longrightarrow C$ be a Boolean epimorphism with the kernel (Ker f) a Boolean algebraic ideal of B. If, for $p \in \operatorname{Ker} f$ and $q \in (\operatorname{Ker} f)'$, $p \wedge q = 0$ then C is a retract of B.

Proof. Under the hypothesis, the exact sequence

$$\operatorname{Ker} f \rightarrowtail B \overset{f}{\twoheadrightarrow} C$$

is split. Hence there exist a Boolean monomorphism h: $C \rightarrowtail B$ such that $f \cdot h = 1_C$. This implies that C is a retract of B.

References

1. P. R. Halmos, *Lectures on Boolean Algebras*, D. Van Nostrand Company, Princeton, (1963).
2. N. Jacobson, *Lectures in Abstract Algebra* I, D. Van Nostrand Company, Princeton, (1951).
3. S. MacLane, *Homology*, Springer, New York (1963)

Han Yang University