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A note on certain integrals related with the generalized hypergeometric function

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1. Introduction. The integrals of type

$$\Gamma_k=\int_0^k (1-\phi^2)^{\beta-1}(k^2-\phi^2)^{\gamma-1}(l^2-\phi^2)^{\delta-1}\phi^{2(3-\beta-\gamma-\delta)}d\phi, \tag{1.1}$$

$$\Gamma_1=\int_k^1 (1-\phi^2)^{\beta-1}(\phi^2-k^2)^{\gamma-1}(\phi^2-l^2)^{\delta-1}\phi^{2(3-\beta-\gamma-\delta)}d\phi, \tag{1.2}$$

where $1/2\leqslant\beta<1$, $1<\gamma\leqslant 3/2$, $1\leqslant\delta<3/2$ and $0<l<k<1$, related to mapping of a certain polygonal boundary onto the real axis by Schwarz-Christoffel transformation ([4], [5], [6]), arising from the problem of ship vibration or ship motion, are brought up last year to our attention.

It is immediately obvious that since $\beta-1<0$ and $1<\gamma$, if $\beta=1/2$, $\gamma=3/2$ and $\delta=1$, then above integrals by a simple change of integration variable reduce to a standard integral, i.e., the Legendre elliptic integral of the second kind. Therefore any hope of solution in a closed form by a simple quadrature is not possible.

However, we found that (1.1) and (1.2) can always be expressed as the generalized hypergeometric function, even if any finite number of factors of the above type are involved, and these integrals satisfy the differential equation of a generalized hypergeometric function. We will give the result as the most suitable, managerable form for numerical calculation and suggest the new method for their calculations.

2. The differential equation. We will show here that the above integrals (1.1) and (1.2) are solutions of the differential equation of a generalized hypergeometric function.

By the change of variable $\dfrac{\phi^2-k^2}{1-k^2}=u$, (1.2) reduces to

$$\Gamma_1=\frac{1}{2}(1-k^2)^{\gamma+\beta-1}\int_0^1 u^{\gamma-1}(1-u)^{\beta-1}[(k^2-l^2)+(1-k^2)u]^{\delta-1} \cdot [k^2+(1-k^2)u]^{3/2-\gamma-\delta-\beta}\,du. \tag{2.1}$$

Let us introduce another new variable $t=1-u$. Then (2.1) becomes

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(2.2) $$\Gamma_1=\frac{1}{2}(1-k^2)^{1/2}\int_0^1 t^{\beta-1}(1-t)^{\gamma-1}(a-t)^{\delta-1}(z-t)^{5/2-\gamma-\delta-\beta}dt,$$

where $a=\frac{1-l^2}{1-k^2}$, $z=\frac{1}{1-k^2}$.

Finally we write Γ_1 as the following form

(2.3) $$\Gamma_1=\frac{1}{2}(-1)^{\delta+\gamma}(1-k^2)^{1/2}\int_0^1 t^{\beta-1}(t-1)^{\gamma-1}(t-a)^{\delta-1}(z-t)^{5/2-\gamma-\delta-\beta}dt.$$

Clearly, by choosing, $\alpha'=5/2-\beta$, $\gamma'=\beta'=-\alpha$, we can always write Γ_1, besides some constant factor, as

(2.4) $$\Gamma_1=(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_0^1(t-a)^{\beta+\gamma+\alpha'-1}(t-b)^{\gamma+\alpha'+\beta'-1}\cdot(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}\,dt,$$

where we have chosen in Γ_1, $c=0$, $b=1$.

It is well-known [2] that above expession for Γ_1 is a solution of the generalized hypergeometric differential equation, smiliar to Papperitz's equation.

(2.5) $$\Gamma_1''+\left\{\frac{1+\alpha-\alpha'}{z-a}+\frac{1+\beta-\beta'}{z-b}+\frac{1+\gamma-\gamma'}{z-c}\right\}\Gamma_1'+\frac{(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma+1)z+\Sigma a(\alpha+\beta'+\gamma'-1)\}}{(z-a)(z-b)(z-c)}\Gamma_1=0,$$

where Σ sign signifies the cyclic permutation of letter a by b, c and α by β, γ. We can reduce Γ_k integral also by a similar method to the standard form (2.4), by choosing suitable α', β', γ'.

3. Evaluation of integrals. We ask first the question about the existence of the integral, since the above integrals range such that the first one becomes singular at $\phi=0$ if $\alpha+\beta+\gamma>3$, the second one becomes singular at $\phi=1$, since $\beta<1$ always. Since $3-(\delta+\beta+\gamma)<1/2$, $2(3-\delta-\beta-\gamma)<1$, therefore by the well-known theorem for improper integral of the second kind, *i.e.*, $\int_0^b x^{-p}dx$ converges if $p<1$, the integral Γ_k exists. For the second one, by simple change of variable, we can see easily that the integral exists also.

Consider first the integral of type (2.1). We rewrite (2.1) in the following form.

(3.1) $$\Gamma_1=\frac{1}{2}(1-k^2)^{\beta+\gamma-1}(1-l^2)^{\delta-1}\int_0^1 t^{\beta-1}(1-t)^{\gamma-1}\cdot[1-(1-k^2)t]^{5/2-(\gamma+\delta+\beta)}(1-yt)^{\delta-1}dt,$$

where $1-k^2=y<1$, since $k<1$ always. Therefore (3.1) becomes

(3.2) $$\Gamma_1=\frac{1}{2}(1-k^2)^{\beta+\gamma-1}(1-l^2)^{\delta-1}\int_0^1 t^{\beta-1}(1-t)^{\gamma-1}(1-xt)^{5/2-(\gamma+\delta+\beta)}(1-yt)^{\delta-1}dt,$$

where $1-k^2=x$. Using the fact that $u<1$, $y<1$, Γ_1 becomes

(3.3) $$\Gamma_1=\frac{1}{2}(1-k^2)^{\beta+\gamma-1}(1-l^2)^{\delta-1}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(n+1-\delta)}{\Gamma(1-\delta)}y^n$$
$$\cdot\int_0^1 t^{n+\beta-1}(1-t)^{\gamma-1}(1-xt)^{5/2-(\gamma+\beta+\delta)}dt.$$

By using the well-known integral representation of hypergeometric function:

$$_2F_1(a,b,c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt,$$

valid in case of $c>b>0$, $|x|<1$, Γ_1 reduces to the following expression, $(b=n+\beta$, $c=\gamma+\beta+n)$,

(3.4) $$\Gamma_1=\frac{1}{2}(1-k^2)^{\beta+\gamma-1}(1-l^2)^{\delta-1}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(n+1-\delta)}{\Gamma(1-\delta)}y^n$$
$$\cdot\frac{\Gamma(\gamma+\beta+n)}{\Gamma(n+\beta)\Gamma(\gamma)}{}_2F_1(\gamma+\delta+\beta-5/2,n+\beta,\gamma+\beta+n;x),$$

valid since $|x|<1$ and $c>b>0$ by the definition of c, b and x. In case of $\delta=1$, (3.4) reduces immediately to

$$\Gamma_1=\frac{1}{2}(1-k^2)^{\beta+\gamma-1}B(\gamma,\beta)_2F_1(\gamma+\beta-3/2,\beta,\gamma+\beta;x).$$

It is now clear why we called the Γ_1 integral as the generalized hypergeometric function. Finally Γ_1 may be written as the double infinite series as follow:

(3.5) $$\Gamma_1=\frac{1}{2}(1-k^2)^{\beta+\gamma-1}(1-l^2)^{\delta-1}\frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma+\delta-5/2)\Gamma(1-\delta)}$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Gamma(m+\gamma+\delta+\beta-5/2)\Gamma(m+n+\beta)\Gamma(n+1-\delta)\frac{x^m y^n}{m!\,n!}.$$

This series converges absolutely for all ranges of parameter, $0<l<k<1$, its evaluation becomes the matter of routine, if we have the good tables of hypergeometric function and Γ-function. However the rate of convergence depends on the value of k. If k is large and approximately equal to one, then it is most rapid. In case of $k\approx 0$, the rate of convergence is very slow. Hence we must devise another method. We will show that in this case, if we expand the integral in a power series of k, then the rate of convergence will be very rapid. For simplicity, we show this method only for the case when $\delta=1$. The procedure works in its complete generality for $\delta\neq 1$. Take, for example, the integral of type Γ_k (1.1). If $\delta=1$, (1.1) reduces to

(3.6) $$\Gamma_k=\int_0^k(1-\phi^2)^{\beta-1}(k^2-\phi^2)^{\gamma-1}\phi^{2(2-\beta-\gamma)}d\phi.$$

By the change of variable $(\phi/k)^2=t$, it becomes

$$(3.7)\qquad \Gamma_k\frac{1}{2}k^{3-2\beta}\int_0^1 t^{3/2-\beta-\gamma}(1-t)^{\gamma-1}(1-k^2t)^{\beta-1}dt.$$

The expression under the integral sign is the well-known integral representation of hypergeometric function. Therefore (3.7) reduces immediately

$$\Gamma_k=\frac{1}{2}k^{3-2\beta}\frac{1}{\Gamma(1-\beta)}\sum_{n=0}^{\infty}\Gamma(1-\beta+n)\ B(5/2-\beta-\gamma+n,\gamma)\frac{k^{2n}}{n!}.$$

This series converges absolutely for all ranges of parameters, and the rate of convergence will be very fast, since k is very small.

The expression for the generalized hypergeometric series (3.5) with more than three parameters may not be suitable for a practical calculation, if the number of parameters or the number of factors of the types in (1.1) and (1.2) increases more than four. The best method for evaluating the integrals (1.1) and (1.2) might be : 1) first find the differential equation satisfied by these integrals. 2) then solve these differential equations directly by the standard power series method.

The complete, more detailed discussion for the related integrals, arising from the conformal mapping of arbitrary polygonal or arcwise boundary onto the real axis, will be discussed elsewhere.

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