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《Original》

Computational Method of Fuel Optimal Control in Regulator System

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Abstract

Determination of a two-point boundary value problem is the key of finding the control function $u(t)$ with the application of the fundamental idea of Minimum principle. The late development shows the discovery of the initial costate vector for the solution of a two-point value problem. As a new technique of determining the optimal control function, Newton's sequential method is examined in this paper.

요 약

최대치(또는 최소치)원리의 기본개념을 써서 제어함수 $u(t)$를 찾는데는 2점간 경계치문제의 해결이 요구된다. 그런데 최근에는 2점간 문제를 푸는 방법으로 초기 costate vector를 찾아 문제를 해결하는 여러가지 편법이 개발되고 있다.

여기에서는 최적제어함수를 찾는 새로운 방법의 하나로써 Newton's Sequential 방법을 적용하였다. 그리고 일차적으로 주어진 문제의 수학적인 전개를 시도하였다. 앞으로 이 방법의 물리적인 의의와 공학적인 가치는 Computer에 의해서 여러가지 응용문제를 해결함으로써 밝혀질 것이다.

Introduction

As the methods of determining optimal control function $u(t)$, the method of functional analysis by Krosovski, Minimum principle by Pontryagin, and Dynamic programming by Bellman have been well introduced as typical ones. The application of Minimum principle focuses on the two-point boundary value problem. This solution of boundary value problem is to be the point of determining the optimal function but Minimum principle has its weakness which provides us with only the necessary condition for determining the optimal function.

The lack of the sufficient condition was indicated by Kelly[1)], Gottlieb[2)], Denn[3)], Kurihari[4)], and Durbeck. They furthered the method, in detail, for finding optimal control and presented how to minimize a sample Hamiltonian at each iteration. Newton's method which the author is going to introduce was widely studied by McReynolds & Bryson[5)], Plant[8)], Knudson (see [7)], [(10)], [(11)], [(12)], [(13)], [(14)], [(15)]. However the sequential operation method was not discussed in their papers.

In this paper, the new sequential operation method applied to Newton's method is designed for determining optimal control function $u(t)$. Of course, the application of sequential operation method to Newton's method is based on accelerating the convergence to the value desired. The given minimization problem, this is put into integral equation and reduced to a vector relation, in its development.

The vector relation is replaced by a sequence of approximate vector operator. A method is designed for choosing the sequence so that Newton's method can be applied to it and approached as closely as desired to the true solution. Under a suitable assumption, the main part of this method is proved by the use of the convergence theory by Kantorovich [18].

Finally, the aim of this paper is to develop a new computational method and to consider its application to fuel minimization of the regulator system and another similar problems.

1. Design of the computational methed

The fixed time fuel optimal control of a linear plant to a given state is examined. One of the mathematical development of the computational method for a problem is shown. The problem is presented, reduced to a two-point boundary value problem, changed to integral form, replaced by a sequence of approximate integral equations, and made ready for the numerical solution by the application of Newton's method.

A) Problem

Given; a. A system described by the linear time invariant(vector) differential equation

$$X(t)=Ax(t)+bu(t) \quad (1,1)$$

where

1. Then vector $x(t)$ is the state.
2. The system matrix A is an $n\times n$ constant matrix.
3. The gain matrix b is an $n\times r$ constant matrix.
4. The r vector $u(t)$ is the control.

b. A fixed time interval

$$t\epsilon[O,T] \quad (1,2)$$

c. Initial and terminal boundary conditions on the state vector

$$X(O)=\xi$$

$$X(T)=\theta \quad (1,3)$$

d. The control variable must satisfy a constraint

$$|u(t)|\leq 1 \text{ for all } t \,\epsilon[O,T] \quad (1,4)$$

e. The fuel functional is

$$J(u)=\int_{o}^{T} |u(t)|\, dt. \quad (1,5)$$

Then, it is desired to find a control variable $u^*(t)$ that

a. Satisfies the constraint (1, 4)

b. Transfers the system (1, 1) from the initial state ξ at time $t=O$ to the terminal state θ at time $t=T$,

c. Minimizes the fuel function (1, 5).

B) The two-point boundary value problem

The relations deduced by applying Pontryagin's minimum principle to the problem are summarized below;

Definition; The "deadzone" function dez [·] is defined as follows;

$$u(t)=\text{dez}[w(t)] \quad (1,6)$$

means $u(t)=1$ when $w(t)>1$

$u(t)=0$ when $|w(t)|<1$

$u(t)=-1$ when $w(t)<-1$

The input-output characteristic of the deadzone function is shown in Fig 1.

Let $u^*(t)$, $t\epsilon\, [O,T]$ be the fuel optimal control, the solution of problem, assuming that one exists. Let $X^*(t)$ be the resulting state on the fuel optimal trajectory. Let $P^*(t)$ $t\epsilon[O,T]$ be the corresponding costate vector.

Then the minimum principle yields the relations

$$H(X^*, u^*, P^*,t)=|u^*(t)|+P^*(t)Ax^*(t)+P^*(t)bu^*(t) \quad (1,7)$$

$$\dot{X}^*(t)=\frac{\partial H}{\partial P^*}=AX^*(t)+bu^*(t) \quad (1,8)$$

$$\dot{P}^*(t)=-\frac{\partial H}{\partial X^*}=-A'P^*(t) \quad (1,9)$$

$$X^*(O)=\xi$$

$$X^*(T)=\theta \quad (1,10)$$

where A' is the transpose of A

and the relation

$H(X^*,u^*,P^*,t)\leq H(X^*,u,P^*,t)$ for all u such that $|u|\leq 1$ yield $u^*(t)=-\text{dez}[b'\ p^*(t)]$ (1, 11) from above Eqs. (1, 8)-(1, 11). Determination of π^*, the optimal costate initial condition vector, will be considered equivalent to solution of the

TPBVP (Two point boundary value problem).

C) Integral equaton form

First write the solution of Eq. (1, 9)

$P^*(t)=e^{-A't}\pi^*$ *where* $\pi^*\equiv P^*(O)$

Defiine for convenience

$$q(t)\equiv e^{-At}b \tag{1, 12}$$

Then the optimal control (1, 11) becomes

$$u^*(t)=-\mathrm{dez}[b'e^{-A't}\pi^*]=-\mathrm{dez}[q'(t)\pi^*] \tag{1, 13}$$

The solution for the state Eq. (1, 8) is

$$X^*(t)=e^{At}[\xi+\int_o^t q(t)\ u^*(t)dt]$$

If the terminal boundary condition (1, 3) is applied, then

$$e^{-AT}\theta=\xi-\int_O^T q(t)\ dez[q'(t)\ \pi^*]dt$$

For later use with Newton's method, the operator $T(\pi)$ is defined as;

$$T(\pi)=\xi-e^{-AT}\ \theta-\int_O^T q(t)\ dez[q'(t)\pi]dt \tag{1, 14}$$

The opterator $T(\pi)$ maps one n-dimensional vector into another

$T(\pi);\ R_n\longrightarrow R_n$

The problem is now reduced to finding π^*, the solution vector of the operator equation

$$T(\pi^*)=0 \tag{1, 15}$$

For simplicity, π^* will be refered to as the solution of the operator $T(\pi)$, also, in most of what follows the final state is the origin, so

$$T(\pi)=\xi-\int_O^T q(t)\ dez[q'(t)\pi]dt \tag{1, 16}$$

D) Sequence of approximate equations

A sequence of approximate operator $\{T_k(\pi)\}$ is now introduced to replace the operator $T(\pi)$. The idea is to start with a very simple operator and work up by step toward the exact operator $T(\pi)$. By doing this properly, Newton's method can be guaranteed to converge at each step, so that a workable computational approach results. Two approximations will be introduced; one is a linear term to get the computations started successfully, and the other is a sequence of smooth functions $u_k(.)$ with a parameter η_k, (k=0,…,k_1,) as $\eta_k\rightarrow\infty$, $u_k(\cdot)\rightarrow u^*(\cdot)$ so the idea is to start with a linear approximation ($\eta_0=0$), then to drive the linear part to zero and increase η_k so that the approximate control $u_k(.)$ converges to the optimal control $u^*(.)$ when the optimal control $u^*(q'(t)\pi^*)$ is replaced by u_k the form of the optimal control argument $q'(t)\pi$ will be retained.

Now a linear one for the simplest useful control will be started.

Change 1; First apply a linear control using the control argument $q'(t)\pi$ *yields*

$u_0(q'(t)\pi)=\alpha_0 q'(t)\pi$

Inserting this control into the differential Eq. (1, 1) and applying the given boundary condition leads to the zeroth approximate operator

$$T_0(\pi)=\xi-\int_O^T q(t)\alpha_0 q'(t)\pi dt$$

Let $W(T)$ be the the controllability matrix

$$W(T)=\int_O^T q(t)q'(t)dt$$

Then, $T_0(\pi)=\xi-\alpha_0 W(T)\pi$ (1, 17)

For approximation of the optimal control function, the exponential form $u_k(.)$ can be brought in by a scalar approximation factor η_k. The deadzone function can also be approximated as closely as desired by an analytic function, since the points of discontinuity are excluded.

Change 2; Introduce an approximate control function $u_k(.)$, using the control argument $q'(t)\pi$ yields

$$u_k(q'(t)\pi)=\frac{1}{2}\{tanh[\eta_k(q'(t)\pi+1)]+tanh[\eta_k(q'(t)\pi-1)]\}. \tag{1, 18}$$

A plot of $u_k(t)$ as a function of $q'(t)\pi$ is shown in Fig. 1 for some typical values of η_k. As η_k increases $u_k(t)$ approaches the deadzone function $u^*(t)$.

The general approximate operator uses both of the above changes.

$$T_k(\pi)=\xi-\alpha_k W(T)\pi-\int_O^T q(t)u_k(q'(t)\pi)dt \tag{1, 19}$$

Let the sequence of approximate operator have k, steps

$$0<\eta_1<\eta_2\cdots\cdots<\eta_{k1}<\infty$$
$$\alpha_0>\cdots\cdots>\alpha_{k2}>\alpha_{k2}+1>\cdots=\alpha_{k1}=0 \tag{1, 20}$$

Where $K_2<K_1$.

In Fig. 3 the sum of the two changes is shown for a typical sequence of approximate operator.

Definition 2; Applied sequentially means the solu-

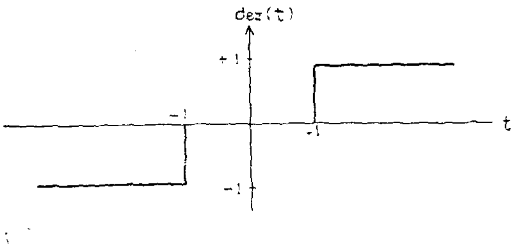

Fig 1 The Deadzone Function Dez

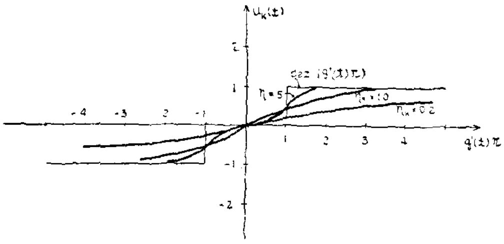

Fig 2 The Approximate Control Function U_k

tion vector π_{k-1} of the previous operator $T_{k-1}(\pi)$ is used as a starting vector for Newton's method on the present operator $T_k(\pi)$.

Properties of the sequence are briefly listed here according to the analytical result.

1. A sequence can be found such that Newton's method converges when applied to each member sequentially.
2. Under suitable restrictions this sequence of operator converges to the exact operator $T(\pi)$.
3. The solutions to the approximate operators lead to sub-optimal control which use only a little more fuel that the optimal control, yet do not require the instantaneous switching of the optimal control.

Definition 3 ; Assume the solution vector π_k of the operator $T_k(\pi)$ has been found. Now make changes $\Delta\eta$ and $\Delta\alpha$ in parameters η and α to form a new operator T_{k+1}. Apply Newton's method to T_{k+1} sequentially (by Def.2) The set of all changess $\Delta\eta$ and $\Delta\alpha$ such that Newton's method converges is called the region of convergence about η_k and α_k in the parameter space. There is a corresponding region of convergence in the space π of solution vectors. Fig. 4 gives some idea of the results. The region of convergence soon includes the exact solution vector π^* as shown in Fig. 4.

As the results

$$\Delta\eta_k=\left(\frac{60}{n^3}\right)^{1/4}\eta_k \tag{1.21}$$

$$\eta_{k+1}=\eta_k+\Delta\eta_k \tag{1, 22}$$

$$\Delta\alpha_k=-\alpha_0\frac{\|g_k\|_1}{\|\xi-e^{-AT}\theta\|_1}(1+\eta_k) \tag{1, 23}$$

$$\alpha_{k+1}=max\{0, \alpha_k+\Delta\alpha_k\} \tag{1, 24}$$

A vector g_k is defined as below,

$$g_k=\int_0^T q(t)u_{k+1}[q'(t)\pi_k]dt$$

E) Applying Newton's method

Newton's method is to be applied to a typical operator $T_k(\pi)$. Given the operator Eq. (1, 19) to find the solution vector π_k such that

$$T_k(\pi_k)=0$$

One linearizes about the current guess

$$T_k(\pi_k)\approx T_k(\pi^i)+(\pi_k-\pi^i)T_k^{(1)}(\pi^i)$$

Then the next iteration is found by solving this linear equation for π_k

$$\pi^{i+1}=\pi^i-[T_k^{(1)}(\pi^i)]^{-1}T_k(\pi^i) \tag{1, 27}$$

Equation (1, 27) is the recursive relation of

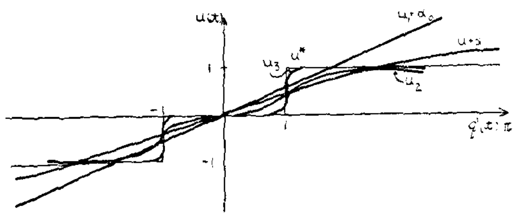

Fig 3 Typical Sequence of Approximate Control

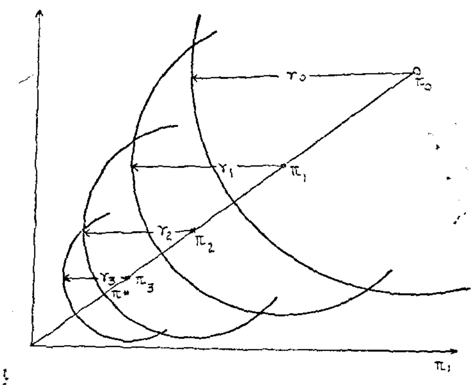

Fig 4. Regions of Convergence for the Sequence (π_k)

Newton's method. Since T_k has vector valued range space, its first derivative is the Jacobian matrix.

$$T_k^{(1)}(\pi) = -\alpha_k W(T) - \int_O^T q(t)q'(t)u_k^{(1)}[q'(t)\pi]dt$$

Then Eq. (1,27) can be written out entirely in matrix notation

$$\pi^{i+1} = \pi^i + \{\alpha_k W(T) + \int_O^T q(t)q'(t)u_k^{(1)}[q'(t)\pi^i]dt\}^{-1}$$
$$\{\xi - \alpha_k W(T)\pi^i - \int_O^T q(t)u_k[q'(t)\pi^i)dt\} \qquad (1,28)$$

From Eq. (1,18), the first derivative of the approximate control function $u_k^{(1)}$ is

$$u_k^{(1)}[q'(t)\pi] = \frac{1}{2}\eta_k\{2 - tanh^2[\eta_k(q'(t)\pi+1)]$$
$$-tanh^2[\eta_k(q'(t)\pi-1)]\} \qquad (1,29)$$

Starting with an initial guess π^0, Eq. (1,28) is applied repeatedly, if at same step, i, $\pi^i \approx \pi^{i-1}$ the inner loop is said to have converged, and the vector π^i is defined to be the solution vector π_k of the operator T_k.

F) Condition for convergence

The sufficient condition for convergence of Newton's method was given by Kantorovich [18]. This sufficient condition is cited in appendix A as the theorem 1.

2. Theoretical Analysis

In this section, the key points are only presented, omitting the detailed development about the method. The suggested method converges to the optimal control which is proved by theorems 1 and 2.

Theorm 1 ;

Given; The operator (1,14) and (1,19)

Assumptions 1) The system matrix of (1,1) is nonsingular i.e. det $A \neq 0$ (2,1)

2) Let the condition

$$|q'(t)\pi^*| = 1 \qquad (2,2)$$

be satisfied at times $t=t_1, t_2, \cdots, t_m$ in the open interval (O,T). The switch times are assumed to be distinct and this will be true if the problem is nonsingular. 3) The necessary and sufficient condition for the uniqueness of π^* given $u^*(t)$, is that the set of vectors $q(t)$, i=1, 2, ⋯, m span the space R_n, that is, the matrix Q must have maximal rank (rank n), where

$$Q = \begin{bmatrix} \vdots & \vdots & & \vdots \\ q(t_1) & q(t_2) & \cdots & q(t_m) \\ \vdots & \vdots & & \vdots \end{bmatrix} \qquad (2,3)$$

Then; for any $\epsilon > 0$ there exists a number $\eta(\epsilon)$ such that for all

$\eta_k > \eta(\epsilon)$

and $\alpha_k = 0 \| T(\pi^*) - T_k(\pi^*) \| < \epsilon$

Proof; Assumptions (2,1) and (2,3) guarantee that the problem is normal and that the argument of the control $q'(t)\pi^*$ does not remain constant for any finite time interval (for a proof of this, see Athems and Falb[8])

Let condition $|q'(t_i)\pi^*| = 1$ occur m times at times

$t_i, \ i = 1, 2, \cdots\cdots, m < \infty$

where $t_i \in [O,T]$

Since $q'(t)\pi^*$ is a continuous function which is never constant, each of the times t_i must be seperated firm its neighbors by some finite amount. Now proceed by removing a small time interval from $[O,T]$ around each of these m points. Let t_{i-} and t_{i+} be the end points of the ith such interval and let B denote the interval. Let B denote the set of all such intervals

$B = \{t; t \in [t_{i-1}, t_{i+1}], \ i = 1, \ 2, \cdots\cdots, m\}$

The end points $t_{i\pm}$ are to be chosen such that

$|q'(t_{i\pm 1})\pi^*| - 1 = \pm 1/\sqrt{\eta}$

Subject to the condition

$B \subset [O,T]$

For small values of η two or more of the intervals may overlap. As η increases the continuity of $q'(t)\pi^*$ guarantees that all the intervals will be seperate for some finite value of η.

The point is that by subdividing the tme interval $[O,T]$ the difference between the exact and approximate operator at π^* can be bounded

$$T(\pi^*) - T_k(\pi^*) = -\int_O^T q(t)\{dez[q'(t)\pi^*] - u_k[q'(t)\pi^*]\}dt$$

Fig. 5 shows that the difference between the dez $[t]$ and u_k $[t]$ function for a given η_k increases as the ± 1 points are approached. So outside the set B the errors increase toward the times $t_{i\pm}$, and are largest where $|q'(t)\pi^*| = \pm 1/\sqrt{\eta}$. Inside the set B it is accurate enough to bound the difference by (1, 0)

Splitting the integral in operator x taking the norm and simplifying yield

$$T(\pi^*) - T_k(\pi^*) \leq \int_B |q(t)|\, dt + \int_{[0,T]-B} |q(t)|\, dt$$

$$[1 - \frac{1}{2} tan(2\eta - \sqrt{\eta}) - \frac{1}{2} tanh\sqrt{\eta}]$$

$$||T(\pi^*) - T_k(\pi^*)|| \leq \int_B |q(t) dt + \int_{[0,T]-B} |q(t)$$

$$|dt[1 - tanh\sqrt{\eta}|$$

Or by regrouping terms

$$||T(\pi^*) - T_k(\pi^*)|| \leq ||C[1 - tanh\sqrt{\eta}] + \int_B |q(t)|$$

$$dt\ tanh\ \sqrt{\eta}||$$

Note; $C = ||\int_0^T |q(t) dt||$ constant

Since $\lim_{\eta \to \infty} \tanh \sqrt{\eta} = 1$

And $\lim_{\eta \to \infty} B = \sum_{i=1}^{m} t_i$ (i.e. of measure zero)

Then both terms of Eq. xx can be made as small as desired by increasing η.

It seems to be reasonable that if $T_k(\pi)$ 0 is close to $T(\pi)$ as in theorem 1, π_k is close to π^* as in theorem 2.

Theorem 2,

Given; 1) The operator equation (1, 14) and (1, 19) of problem

2) Assumption (2, 1), (2, 2), and (2, 3)

Then; for any $\epsilon_1 > 0$ there exists a number $\eta(\epsilon)$ such that whenever

$\eta_k > \eta(\epsilon)$

then

$||\pi^* - \pi_k|| < \epsilon_1$

3. Conclusion

The original minimization problem for a regulator, (The fuel optimal control such as for reactor, internal combustion engines, chemical process, and boiler) was converted to a two-point boundary value problem. This was put into integral form and reduced to a nonlinear vector relation (or operator): i.e., the problem can be solved once the initial costate vector π^* is found. The vector relation was replaced by a sequence of approximate vector operator. A method was designed for choosing the sequence so that Newton's method could be applied to it and approached as closely as desired to the true solution.

The procedure consists of applying Newton's method sequentially to the sequence of approximate operator to determine their solution vector $\{\pi_k\}$. These vectors lead to a sequence of approximate control which converges to the optimal control.

Theorems 1 and 2 are theoretically shown under a suitable assumption that convergence of this method is going to approach the true solution.

Finally for the digital computer application to various engineering problems, further study will have to be followed.

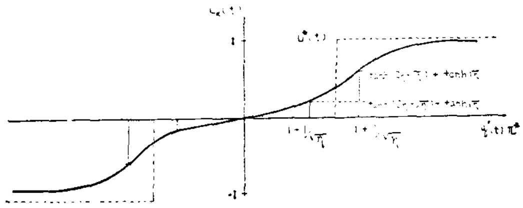

Fig 5 Partition of Approximate Control U_K for Theorem 1

Appendix A;

Theorem 1 ; Suppose that,

1) The second derivative operator $P^{(2)}[y]$ exists and is continuous on the set Y_1.

2) The first derivative inverse operator Γ_0 exists

3) $||\Gamma_0 P(y^0)|| \leq \beta_0$

4) $||\Gamma_0 P^{(2)}(y)|| \leq \beta$ *For all* $y \epsilon Y$

5) $h = \beta\beta_0 \leq 1/2$

6) $Y_1 \supset S_{r0}(y^0)$

where $r_0 = \frac{1 - \sqrt{1 - 2h}}{\beta}$

Then 1) There is a solution $Y^* \in S_{r'}(y')$ such that $P(Y^*) = 0$

2) This solution is unique in the set

$Y_1 \cap S_{r1}(y^0)$ where $r_1 = \frac{1 + \sqrt{1 - 2h}}{\beta}$

3) Newton's method converges to the solution Y^*.

4) The rate of convergence is characterized by inequality

$$||y^* - y^i|| \leq \frac{(2h)^{2i}}{\beta^{2i}}$$

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