CERTAIN RULES FOR TWO SIDED LAPLACE TRANSFORMATIONS

By R. S. Dahiya

1. In this paper the auther has obtained certain rules on bilateral Laplace transformation. These rules are next applied to evaluate certain operational relations in two variables. The domain of convergence has been fully investigated by P.C. Chatterjee [1] and R.K. Gupta [2] in the Bullitin of Calcutta Mathematical Society and in their theses for doctor's degrees. So the author has made no attempt here to discuss the region of convergence.

2. Let $f(x, y)$ be a function of two real variables x and y defined for all pairs (x, y) in $-\infty<x<\infty$ and $-\infty<y<\infty$ and integrable in every finite rectangle $R_{xy}: -X<x<X, \ -Y<y<Y$.

If for a pair of complex quantities p and q the limit of each of the following double integrals

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \int_0^x \int_0^y e^{-px-qy} f(x, y)\, dx\, dy \tag{2.1}$$

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \int_0^x \int_{-y}^0 e^{-px-qy} f(x, y)\, dx\, dy \tag{2.2}$$

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \int_{-x}^0 \int_0^y e^{-px-qy} f(x, y)\, dx\, dy \tag{2.3}$$

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \int_{-x}^0 \int_{-y}^0 e^{-px-qy} f(x, y)\, dx\, dy \tag{2.4}$$

exists and then evidently the double integral

$$L_\pi^2\{f\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-px-qy} f(x, y)\, dx\, dy \tag{2.5}$$

exists and we call it the two-dimensional bilateral Laplace integral of $f(x, y)$ for the pair of values p and q. If $L_\pi^2\{f\}$ exists not only for a particular pair of values p and q but also for every pair of values (p, q) in certain associated regions of the complex p and q planes, we call it the two dimensional bilateral Laplace transform of $f(x, y)$. We write

$$F(p, q)=pq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-px-qy} f(x, y)\, dx\, dy \tag{2.6}$$

as the definition integral and symboliically denote it as

$$f(x, y) \underset{\pi}{\fallingdotseq} F(p, q) \text{ or } F(p, q) \underset{\pi}{\fallingdotseq} f(x, y) \tag{2.7}$$

3. Rule 1 : *Let*

(i) $f(p) \doteqdot F(x)$, (ii) $K(p) \doteqdot f(x)$,

$$\text{then } \frac{2pq}{p^2-q^2} K\left(\frac{p-q}{p+q}\right) \doteqdot F\left(\frac{x+y}{x-y}\right). \tag{3.1}$$

PROOF : Consider the integral

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-px-qy} F\left(\frac{x+y}{x-y}\right) dx\, dy.$$

On putting $x+y=u$; $x-y$, we get

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-p(u+v)/2-q(u-v)/2} F(u/v)\, dudv$$

On writing uv for u, we obtain

$$I=\frac{1}{2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(p-q)v}\, v\, dv \int_{-\infty}^{\infty} e^{-\frac{1}{2}(p+q)uv} F(u)du$$

$$=\frac{1}{p+q}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(p-q)v} f[\tfrac{1}{2}(p+q)v]\ \ dv$$

$$=\frac{2}{(p+q)^2}\int_{-\infty}^{\infty} e^{-\frac{p-q}{p+q}} f(v)dv=\frac{2}{(p+q)(p-q)} K\left(\frac{p-q}{p+q}\right).$$

$$\therefore\ \frac{2pq}{(p^2-q^2)} K\left(\frac{p-q}{p+q}\right) \doteqdot F\left(\frac{x+y}{x-y}\right).$$

RULE 2 : *Let* (i) $f(p) \doteqdot F(x)$, (ii) $K(p) \doteqdot f(1/x)$,

$$\text{then } \frac{2pq}{p^2-q^2} K\left(\frac{p^2-q^2}{4}\right) \doteqdot F(x^2-y^2). \tag{3.2}$$

RULE 3 : *Let* (i) $\phi(p) \doteqdot F(x)$, (ii) $K(p) \doteqdot \phi(x)$,

$$\text{then } \frac{p}{p-q} K\left(\frac{p-q}{q}\right) \doteqdot F\left(\frac{x+y}{x}\right). \tag{3.3}$$

RULE 4 : *Let* (i) $\phi(p) \doteqdot f(x)$, (ii) $K(p) \doteqdot F(x)$,

$$\text{then } \frac{p}{p-q}\phi(p-q)K(q) \doteqdot f(x)\ F(x+y). \tag{3.4}$$

RULE 5 : *Let* (i) $\phi(p) \doteqdot f(x)$, (ii) $K(p) \doteqdot F(x)$,

$$\text{then } \frac{2pq}{p^2-q^2}\ \frac{\phi(p-q)}{2}\ K\frac{(p+q)}{2} \doteqdot f(x-y)\ F(x+y). \tag{3.5}$$

3. APPLICATIONS:

(a) Consider $\phi(p)=\sqrt{\pi}\ pe^{p^2/4} \doteqdot e^{-x^2}=f(x)$; $-\infty<R(p)<\infty$,

and $K(p)=\sqrt{\pi}\, p^{n+1}\, e^{p^2/4} \doteqdot e^{-x^2}\, He_n(-x)=F(x);\ -\infty<R(p)<\infty.$
Then from (3.5), we get

$$\frac{\pi pq}{2}\,\frac{(p+q)^n}{2^n}\,\exp\Big[\frac{(p-q)^2}{16}+\frac{(p+q)^2}{16}\Big] \doteqdot \exp\left[-(x-y)^2-(x+y)^2\right]\, He_n(-x-y),$$
$$-\infty<R(p\pm q)<\infty.$$

(b) $\phi(p)=\Gamma(p+1)K_{v-p}(\mathrm{a}) \doteqdot \dfrac{K_v\left(\sqrt{a^2+2ae^{-x}}\right)}{\left(1+\frac{2}{a}e^{-x}\right)^{v/2}}=f(x);\ 0<R(p)<\infty.$

$$F(x)=k_0\left[\sqrt{a^2+b^2+2ab\cosh x}\right] \doteqdot 2p\, k_p(a)k_p(b);\ -\infty<R(p)<\infty,\ R(a,b)>0.$$

Hence from (3.5), we get

$$pq\frac{\Gamma(p-q)}{2}\, k_{v-p/2-q/2}(a)k_{(p+q)/2}(a)k_{(p+q)/2}(b) \doteqdot \frac{k_v\left[\sqrt{a^2+2ae^{y-x}}\right]}{\left(1+\frac{2}{a}e^{y-x}\right)^{v/2}}$$
$$\times k_0\left[\sqrt{a^2+b^2+2ab\cosh(x+y)}\right],\ 0<R(p-q)<\infty,\ -\infty<(p+q)<\infty,$$
$$R(a,b)>0.$$

(c) $\phi(p)=p\dfrac{\Gamma(a+c)\,\Gamma(p+a)\,\Gamma(b-p)}{\Gamma(a+b)\,\Gamma(c-p)} \doteqdot \exp\Big[\frac{1}{2}(c-a)x-\frac{1}{2}(e^{-x})\Big]$

$$\times M_{b+\frac{1}{2}(a-c),\frac{1}{2}(a+c-1)}(e^{-x})=f(x),\ \ -R(a)<R(p)<R(b)$$

$$K(p)=\frac{p\Gamma(p+1)\,\Gamma(p+a)\,\Gamma(b-p)}{\Gamma(a+b)\,\Gamma(b+1)} \doteqdot \exp\Big[\frac{1}{2}e^{-x}-ax\Big]W_{-\frac{1}{2}a-b,\ \frac{1}{2}(1-a)}(e^{-x}),$$
$$-R(a,1)<R(p)<R(b).$$

Hence from (3.5), we get

$$\frac{pq\,\Gamma\left(\frac{1}{2}(p-q)+a\right)\Gamma\left(\frac{1}{2}(p-p)+b\right)\Gamma\left(\frac{1}{2}(p+q)+1\right)\Gamma\left(\frac{1}{2}(p+q)+a\right)\Gamma\left(b-\frac{1}{2}(p+q)\right)\Gamma(a+c)}{2[\Gamma(a+b)]^2\ \Gamma\left[c-\frac{1}{2}(p-q)\right]\Gamma(b+1)}$$

$$\doteqdot \exp\Big[\frac{1}{2}\left\{(c-a)(x-y)-e^{-(x-y)}\right\}\Big]\times\exp\Big[\frac{1}{2}e^{-x-y}-a(x+y)\Big]$$

$$\times M_{b+\frac{1}{2}(a-c),\frac{1}{2}(a+c-1)}(e^{y-x})\ W_{-\frac{1}{2}a-b,\ \frac{1}{2}(1-a)}(e^{-x-y}),$$

$$-R(a)<R\Big(\frac{1}{2}p-\frac{1}{2}q\Big)<R(b),\ -R(a,1)<R(\frac{1}{2}p+\frac{1}{2}q)<R(b).$$

(d) Consider $\phi(p)=\dfrac{\Gamma(v)}{p^{v-1}}\ {}_{r+1}F_s(\alpha,\ v;y;-\frac{1}{p}) \doteqdot t^{v-1}\ {}_rF_s(\alpha;y;-t)U(t),$

$$0<R(p)<\infty,\ r<s,\ R(v)>0;$$

and $K(p)=2p\, k_p(a) \doteqdot e^{-a\cosh x};\ -\infty<R(p)<\infty,\ R(a)>0.$

Hence from (3.5), we obtain

$$pq\,(p-q)^{-v}\,\Gamma(v)\,2^{v}{}_{r+1}F_{s}\Big(\alpha,\ v;\ r;\ -\frac{2}{p-q}\Big)\,k_{\frac{1}{2}(p+q)}(a) \doteqdot (x-y)^{v-1}$$

$$\times{}_rF_s(\alpha;\ r;\ -x+y)\ U(x-y)e^{-a\cosh\,(x+y)},\quad 0<R(p-q)<\infty,$$

$$-\infty<R(p+q)<\infty,\ R(a)>0,\ R(v)>0,\ r<s.$$

(e) Consider $\phi(p)=\sqrt{\pi}\,pe^{p^2}\ \mathrm{Erfc}(p) \doteqdot e^{-x^2/4}U(x)\equiv f(x);\ -\infty<R(p)<\infty.$

$$K(p)=\frac{p\Gamma(2p+1)\,\Gamma\Big(\frac{1}{2}\mu+\frac{1}{2}v-p\Big)}{\Gamma\Big(\frac{1}{2}\mu+\frac{1}{2}\nu+1+p\Big)\,\Gamma\Big(\frac{1}{2}\mu-\frac{1}{2}\nu+1+p\Big)\Gamma\Big(\frac{1}{2}\nu-\frac{1}{2}\mu+1+p\Big)}$$

$$\doteqdot J_{\mu}(2e^{\frac{1}{2}x})\,J_{\nu}\,(2e^{\frac{1}{2}x});\ -1/2<R(p)<1/2R(\mu+v).$$

Hence from (3.5), we obtain

$$\frac{\sqrt{\pi}\,pq\,e^{\frac{(p-q)^2}{2}}\ \mathrm{Erfc}\Big(\frac{1}{2}p-\frac{1}{2}q\Big)\,\Gamma(p+q+1)}{2\Gamma\Big(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2}q+1\Big)}$$

$$\times\frac{\Gamma\Big(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}p-\frac{1}{2}q\Big)}{\Gamma\Big(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2}q+1\Big)\,\Gamma\Big(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}p+\frac{1}{2}q+1\Big)}$$

$$\doteqdot e^{-\left(\frac{x-y}{2}\right)^2}\times J_{\mu}(2e^{\frac{1}{2}(x+y)})\ J_{\nu}\,(2\ e^{\frac{1}{2}(x+y)})\ U(x-y),$$

$$-\infty<R\,[(p)-(q)]<\infty,\ -1<R(p+q)<R(\mu+\nu).$$

(f) Consider $\phi(p)=-\dfrac{p\pi}{\sin(\pi p)} \doteqdot (e^x+1)^{-1}=f(x),\ -1<R(p)<0,$

and $K(p)=\pi p\ \cot\ \pi p \doteqdot (e^x-1)^{-1}=F(x);\ -1<R(p)<0.$

Then from (3.4), we get

$$-\,[\pi^2 pq\cot(\pi p)]\,/\sin[\pi(p-q)] \doteqdot (e^x+1)^{-1}(e^{x+y}+1)^{-1},$$

$$-1<R(q)<0,\ -1<R\ (p-q)<0.$$

(g) Consider $\phi(p)=2pk_p(a) \doteqdot e^{-a\cosh\,x}\equiv f(x);\ -\infty<R(p)<\infty,$

and $K(p)=\frac{1}{2}p\,\Gamma\Big(p+\frac{1}{2}v\Big)\Gamma\Big(p-\frac{1}{2}v\Big) \doteqdot k_v(2e^{-x/2})\equiv F(x)$

$$1/2|R(v)|<R(p)<\infty.$$

Hence from (3.4), we get

$$pq\,k_{p-q}(a)\,\Gamma\Big(q+\frac{1}{2}v\Big)\ \Gamma\Big(q-\frac{1}{2}v\Big) \doteqdot \exp(-a\cosh x)k_{\ v}\Big[2e^{-\frac{1}{2}(x+y)}\Big]$$

$$-\infty<R(p-q)<\infty,\ 1/2|R(v)|<R(q)<\infty.$$

(h) Consider $\phi(p)=2pk_p(c) \doteqdot e^{-c\ \cosh x}\equiv f(x);\ -\infty<R(p)<\infty,$

and $K(p)=2p\,k_p(a)k_p(b) \doteqdot k_0\big[\sqrt{a^2+b^2+2ab\cosh\ x}\,\big]\equiv F(x),\ -\infty<R(p)<\infty.$

Then from (3.4), we obtain

$$4pq\, k_{p-q}(c)\, k_q(a)\, k_q(b) \doteqdot e^{-c\cosh x} k_0\left[\sqrt{a^2+b^2+2ab\cosh(x+y)}\right],$$
$$-\infty < R(p-q) < \infty,\quad -\infty < R(q) < \infty.$$

(i) Consider $\phi(p) = p\,\Gamma(p+a)\,\Gamma(p-a) \doteqdot 2k_{2a}(2e^{-\frac{1}{2}x}) \equiv f(x);\ R(a) < R(p) < \infty,$

and $K(p) = \Gamma(p+1) \doteqdot \exp(-e^{-x}) \equiv F(x);\ 0 < R(p) < \infty.$

Then from (3.4), we obtain

$$p\,\Gamma(q+1)\,\Gamma(p-q+a)\,\Gamma(p-q+a) \doteqdot 2e^{-e^{-x-y}}\, k_{2a}(2e^{-\frac{1}{2}x});$$
$$R(a) < R(p-q) < \infty,\ 0 < R(q) < \infty.$$

(j) Consider $\phi(p) = \Gamma(p+1)\sin\left(\frac{\pi}{2}p\right) \doteqdot \sin(e^{-x}) \equiv f(x);\ -1 < R(p) < 1,$

and $K(p) = \Gamma(p+1)\cos\left(\frac{\pi}{2}p\right) \doteqdot \cos(e^{-x}) \equiv F(x);\ 0 < R(p) < 1.$

Hence from(3.4), we obtain

$$p(p-q)^{-1}\,\Gamma(p-q+1)\,\Gamma(q+1)\ \sin\left[\frac{\pi}{2}(p-q)\right]\ \cos\left(\frac{\pi}{2}q\right) \doteqdot \sin(e^{-x})\cos(e^{-x-y}),$$
$$-1 < R(p-q) < 1,\ 0 < R(q) < 1.$$

(k) Consider $\phi(p) = \Gamma(p+1)\sec(\Pi p) \doteqdot \exp(-e^{-x})\,\mathrm{Erfc}(e^{-\frac{1}{2}e^x}) \equiv f(x),$

$0 < R(P) < \frac{1}{2}$, and $K(p) = \pi\,\mathrm{cosec}(\pi p) \doteqdot \log(e^x+1) \equiv F(x),\ 0 < R(p) < 1.$

Hence from (3.4), we obtain

$$\pi p(p-q)^{-1}\,\Gamma(p-q+1)\,\mathrm{cosec}(\pi q)\sec[\pi(p-q)] \doteqdot \exp(-e^{-x})\,\mathrm{Erfc}(e^{-1/2x}$$
$$\times\log(e^{x+y}+1),\qquad 0 < R(p-q) < \frac{1}{2},\ 0 < R(q) < 1.$$

(l) Consider

$$\phi(p) = p\frac{\Gamma(a-p)}{\Gamma(b+p)} \doteqdot e^{\frac{1}{2}(a-b+1)x}\, J_{a+b-1}(2e^{\frac{1}{2}x}) \equiv f(x);$$
$$\frac{1}{2}R(a-b) - \frac{1}{4} < R(p) < R(a),$$

and $K(p) = \Gamma(p+1)\,\Gamma(a-p)\,[\Gamma(a+p)]^{-1} \doteqdot \sqrt{\pi}\,\exp\left(\frac{1}{2}x - e^{\sqrt{x}}\right) I_{a-\frac{1}{2}}\left(\frac{1}{2}e^x\right);$

$$0 < R(p) < R(a).$$

Hence from (3.4), we obtain

$$p\,\Gamma(a-p+q)\,\Gamma(q+1)\,\Gamma(a-p)\,\Gamma(b+p-q)\,\Gamma(a+q) \doteqdot \sqrt{\pi}\exp\left[\frac{1}{2}(a-b+1)x\right.$$
$$\left.+\frac{1}{2}(x+y) - \frac{1}{2}e^{x+y}\right] J_{a+b-1}(2e^{\frac{1}{2}x})\, I_{a-\frac{1}{2}}\left(\frac{1}{2}e^{x+y}\right);$$
$$\frac{1}{2}R(a-b) - \frac{1}{4} < R(p-q) < R(a),\ 0 < R(q) < R(a).$$

In a similar manner, we can calculate many correspondences in two variables.

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