ON BILATERAL CALCULUS IN TWO VARIABLES II

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1. This paper is a continuation of my paper I, in the Kyungpook Mathematical Journal. I have slightly modified the theorem in a way that the 'original' and the 'image' have been interchanged. The proof of the theorem is on the same line and I state the theorem in the following way.

2. THEOREM 1. *Let*

(i) $F(p_1, p_2) \doteqdot G(s_1, s_2)$, $1/p_r \doteqdot s_r$ $(r=1,2)$; *where* $L^2_\pi\{G\}$ *is absolutely convergent in a pair of associated convergence domains* S_{p_1} *and* S_{p_2} *which may be the half-planes* $R(p_i)>0$, *or* $-\infty<R(p_i)<\infty$.

(ii) $\phi_r(p_r, s_r) \doteqdot \exp\left[-s_r\theta_r^k(x_r)+\dfrac{1}{\theta_r^1(x_r)}\right]\phi_r(x_r)$, $r=1, 2$ *valid in a pair of associated convergence domains* D_{p_1} *and* D_{p_2} *which also may be the half-planes* $R(p_i)>0$, $(1/p_r \doteqdot x_r$, $r=1, 2)$ *and* s_1, s_2 *are real parameters.*

(iii) $\theta_r^k(x_r)+\theta_r^{-1}(x_r) \in s_{p_r}$, $r=1,2$.

(iv) $\phi_r(p_r, s_r)$ *is bounded and absolutely integrable in* $(-\infty, \infty)$.

Then
$$g(p_1, p_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_1(p_1, s_1)\ \phi_2(p_2, s_2)\ G(s_1, s_2)\ ds_1 ds_2$$

$$\doteqdot f(x_1, x_2)=\frac{F\left(\theta_1^k(x_1)+\dfrac{1}{\theta_1^l(x_1)},\ \theta_2^k(x_2)+\dfrac{1}{\theta_2^l(x_2)}\right)}{\left[\theta_1^k(x_1)+\dfrac{1}{\theta_1^l(x_1)}\right]\left[\theta_2^k(x_2)+\dfrac{1}{\theta_2^l(x_2)}\right]}\phi_1(x_1)\ \phi_2(x_2).$$

3. COROLLARIES

(*a*): Let (i) $F(p_1, p_2) \doteqdot G(s_1, s_2)$, valid in a pair of associated half-planes H_{p_1} and H_{p_2}.

(ii) Let $k=1$, $l=-1$ and $\phi_r(x_r)=1$, $r=1, 2$.

(iii) $\theta_r(x_r)=\cosh(x_r)+a_r x_r>0$ for all values of x_r in $(-\infty, \infty)$.

Since $\exp(-2s_r\ \cosh x_r\ -2a_r s_r x_r) \doteqdot 2p_r\ K_{p_r+2a_r s_r}\ (2s_r) \doteqdot \phi_r(p_r,\ s_r)$

Then $4p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{p_1+2a_1s_1}(2s_1)\ K_{p_2+2a_2s_2}(2s_2)\ G(s_1, s_2)\ ds_1ds_2$

$$\doteqdot f(x_1, x_2)=\frac{F\left[2(\cosh x_1+a_1x_1),\ \ 2(\cosh x_2+a_2x_2)\right]}{4(\cosh x_1+a_1x_1)\ (\cosh x_2+a_2x_2)}$$

provided that $L_\pi^2\{f\}$ is absolutely convergent in a pair of associated domains D_{p_1} and D_{p_2} where D_{p_r} denotes the region $-\infty<R(p_r)<\infty$.

Cor. (b): Let (i) $\phi_r(x_r)=U(x_r)$, $r=1, 2$.

(ii) $\theta_r(x_r)=x_r/2$, $r=1, 2$ and $k=1$ and $l=-1$.

$$\exp(-s_r x_r)U(x_r) \doteqdot p_r/(p_r+s_r)=\phi_r(p_r, s_r), \quad -R(s_r)<R(p_r)<\infty, \quad (r=1, 2).$$

Hence from the theorem, we get

$$p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{G(s_1, s_2)\,ds_1\,ds_2}{(p_1+s_1)(p_2+s_2)} \doteqdot \frac{F(x_1, x_2)}{x_1x_2}U(x_1)\,U(x_2).$$

Cor. (c): Let (i) $\theta_r(x_r)=\exp(x_r)$, $k=1$, $l=-1$

and $\phi_r(x_r)=[1-\exp(-x_r)]^{v-1}\,U(x_r)$.

$$\exp(-2s_r e^{x_r})\,(1-e^{-x_r})^{v-1}U(x_r) \doteqdot \Gamma(v)\,p_r e^{-s_r}\,(2s_r)^{\frac{1}{2}(p_r-1)}$$

$$\cdot\, W_{1/2-p_r/2-v,\ -p_r/2}(2s_r), \quad R(s_r, v)>0.$$

Hence from the theorem, we get

$$[\Gamma(v)]^2 p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-s_1-s_2)(2s_1)^{(p_1-1)/2}\,(2s_2)^{(p_2-1)/2}\,W_{1/2-p_1/2-v,\ -p_1/2}(2s_1)$$

$$\times\, W_{1/2-p_2/2-v,\ -p_2/2}(2s_2)\,G(s_1, s_2)ds_1ds_2$$

$$\doteqdot 1/4\,\exp(-x_1-x_2)\,[(1-e^{-x_1})\,(1-e^{-x_2})]^{v-1}\,F(2e^{x_1}, 2e^{x_2})\,U(x_1)\,U(x_2).$$

Cor. (d): Let $\theta_r(x_r)=1/(2x_r)$, $k=1$, $l=-1$ and $\phi_r(x_r)=x_r^{v-1}\,U(x_r)$.

$$\exp(-s_r/x_r)x_r^{v-1}\,U(x_r) \doteqdot 2s_r^{v/2}p_r^{1-v/2}\,K_v(2/s_rp_r), \quad R(s_r)>0.$$

Hence from the theorem, we get

$$4(p_1p_2)^{1-v/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_1s_2)^{v/2}\,K_v(2/s_1p_1)\,K_v(2/s_2p_2)\,G(s_1, s_2)\,ds_1ds_2$$

$$\doteqdot (x_1x_2)^v\,F(1/x_1, 1/x_2)\,U(x_1)\,U(x_2).$$

4. EXAMPLES:

(1) Suppose $F(p_1, p_2)=(p_1p_2)^{-1/2}\exp(-1/p_1p_2) \doteqdot (x_1x_2)^{1/6}\,J_{1/2,1/2}(3\sqrt[3]{x_1x_2})U(x_1)$ $\times\, U(x_2) = G(x_1, x_2)$

By the Cor. (d), we get

$$\int_0^{\infty}\int_0^{\infty}(s_1s_2)^{v/2+1/6}K_v(2/s_1p_1)\,K_v(2/s_2p_2)\,J_{1/2,1/2}(3\sqrt[3]{s_1s_2})\,ds_1ds_2$$

$$=\frac{1}{4}\,[\Gamma(v+3/2)]^2\exp(p_1p_2/2)\,p_1^{v/2-1/2}\,p_2^{v/2-1/2}\,W_{-v-1,0}(p_1p_3), \quad R(v)>-3/2.$$

(2) Suppose $F(p_1, p_2)=\log p_1 p_2 \doteqdot [2\phi(1)-\log x_1 x_2]\ U(x_1)\ U(x_2)=G(x_1, x_2)$

By the Cor. (d), we get (when $v=0$)

$$\int_0^\infty \int_0^\infty K_0(2\sqrt{s_1 p_1})\ K_0(2\sqrt{s_2 p_2})\ [2\phi(1)-\log s_1 s_2]\ ds_1\, ds_2=\frac{\log p_1 p_2-2\phi(1)}{4 p_1 p_2}.$$

(3) Suppose $F(p_1, p_2)=\exp(1/p_1 p_2) \doteqdot J_{0,0}(-3\sqrt{x_1 x_2})\ U(x_1)\ U(x_2)=G(x_1, x_2)$.

By the Cor. (d), we get (when $v=0$)

$$\int_0^\infty \int_0^\infty K_0(2\sqrt{s_1 p_1})\ K_0(2\sqrt{s_2 p_2})\ J_{0,0}(-3\sqrt[3]{s_1 s_2})\ ds_1\, ds_2$$
$$=(1/4)\ \exp(-p_1 p_2)\ E_i(p_1 p_2).$$

(4) Suppose $F(p)=\Gamma(n)\ p^{-n-m} \exp(-1/p)\ L_h^m(1/p) \doteqdot t^{n+m/2}\ J_m(2/t)\ U(t)$.

By the Cor. (d), we get

$$2p^{1-v/2}\int_0^\infty s^{n+v/2+m/2}\ J_m(2/s)\ K_v(2/s)\ ds \doteqdot (n)\ x^{v+n+m} \exp(-x)\ L_h^m(x)\ U(x),$$

or $(m+1)^{-1}(n+1)^{-1}\ \Gamma(m+v+n+1)\Gamma(m+n+1)\ p^{-v-m-n}$

$$\times {}_2F_1(m+n+v+1,\ m+n+1;\ m+1; -1/p) \doteqdot x^{v+m+n} \exp(-x) L_h^m(x)\ U(x),$$
$R(m+n+1)>0$, $R(v+m+n+1)>0$.

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