A SHORT PROOF OF ITO'S THEOREM CONCERNING THE POINTWISE ERGODIC THEOREM

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The purpose of this note is to give a short proof of the

THEOREM (Ito [3]). *Let (X, β, m) be a finite measure space. Let T be a positive linear operator of $L^1(m)$ into itself with $\|T\|_1=1$. If the sequence $\{\overline{T}^n 1\}$, $\overline{T}^n = (1/n) \sum_{k=0}^{n-1} T^k$, is uniformly integrable, then the pointwise ergodic theorem holds for T, i.e.,*

$\lim \overline{T^n} f$ *exists m-a.e. (almost everywhere) for each $f \in L^1(m)$.*

In the following we assume the uniform integrability of $\{\overline{T}^n 1\}$. We note that the assumption is equivalent to the weak convergence of $\{\overline{T}^n 1\}$. If we write $h=w\text{—}\lim \overline{T}^n 1$, then $h \in L^1_+(m)$ and $Th=h$. Let C be the conservative part of the space X associated with T. We write $C_g=\{\sum_{k=0}^{\infty} T^k g=\infty\}$ for each $g \in L^1_+(m)$. It follows readily that $\{h>0\}=C_h \subseteq C$. We can also prove

LEMMA. $C=C_h$.

By the general ergodic theorem of Chacon-Ornstein [1], we have the pointwise ergodic theorem for T on the conservative part C. On the other hand, it follows from a theorem of Hopf [2] that for each positive linear operator T of $L^1(m)$. into $L^1(m)$ with $\|T\|_1 \leqq 1$, the pointwise ergodic theorem holds on the dissipative part $D=X-C$. Thus we have the theorem.

PROOF of LEMMA. It is enough to show $m(C \cap A)=0$ where $A=X-C_h$. Let U be the adjoint of T. 1_A denotes the characteristic function of the set A. Since the set C_h is an invariant set, i.e., $U1_{C_h}=1_{C_h}$ on C and $U1_A=0$ on C_h, we readily have $U1_A \leqq 1_A$. However U being positive implies that the sequence $\{U^k 1_A\}$ is decreasing (see also Lemma 3 of [3]). In fact we have $\lim U^k 1_A=0$ m-a.e.

If we put $g=\lim U^k 1_A$, then $g=\lim \overline{U^n} 1_A$ where $\overline{U^n}=(1/n)\sum_{k=0}^{n-1} U^k$. By Lebesgue dominated convergence theorem and the weak convergence assumption. we have

$$\int g=\lim\int \overline{U^n}\ 1_A=\lim \int_A \overline{T^n}1=\int_A h=0.$$

Thus $g=0$ m-$a.e.$

On the other hand, $U^k 1_{C\cap A}\leqq U^k 1_A$, $k=1,2,\cdots$, implies $\lim U^k 1_{C\cap A}=0$ m-$a.$ $e.$ We note that the set $C\cap A$ is invariant [2], i.e., $U1_{C\cap A}=1_{C\cap A}$ on C. By the usual argument we have

$$0=\lim\int U^k 1_{C\cap A}\geqq\lim\int_C U^k 1_{C\cap A}=\int_C 1_{C\cap A}=m(C\cap A)\geqq 0 \text{ and } m(C\cap A)=0.$$

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