NOTE ON INFINITESIMAL η-CONFORMAL AND CL-TRANSFORMATIONS OF SPECIAL CONTACT METRIC SPACES

By U-Hang Ki

Y. Tashiro and S. Tachibana showed some characteristic properties of Fubinian and C-Fubinian manifolds in their paper [1], where the notion of C-loxodromes was introduced in an almost contact manifold with affine connection. Recently H. Mizusawa defined an infinitesimal η-conformal transformation in a contact metric space [2]. K. Takamatsu and H. Mizusawa have shown some relations in a compact normal contact metric space under an infinitesimal CL-transformation [3].

In the previous paper [4], We have obtained that an infinitesimal CL-transformation in a normal contact and K-contact metric space had some analogous properties of [3]. In this paper, we study on infinitesimal η-conformal and CL-transformations in K-contact and normal contact metric spaces.

§1. Preliminaries

An n $(=2m+1)$-dimensional differentiable manifold M of class C^∞ with (φ, ξ, η, g)-structure (or an almost contact metric structure) has been defined by S. Sasaki [5]. By definition it is a manifold with tensor fields $\varphi_j^{\ i}$, ξ^i, η_i and so called an associated Riemannian metric tensor g_{ji} defined over M which satisfy the following relations:

$$\xi^i\eta_i=1, \tag{1.1}$$

$$\text{rank } |\varphi_j^{\ i}|=n-1, \tag{1.2}$$

$$\varphi_j^{\ i}\xi^j=0, \tag{1.3}$$

$$\varphi_j^{\ i}\eta_i=0, \tag{1.4}$$

$$\varphi_j^{\ r}\varphi_r^{\ i}=-\delta_j^i+\xi^i\eta_j, \tag{1.5}$$

$$g_{ji}\xi^j=\eta_i, \tag{1.6}$$

$$g_{ji}\varphi_h^{\ j}\varphi_k^{\ i}=g_{kh}-\eta_k\eta_h\ . \tag{1.7}$$

On the other hand let M be a differentiable manifold with a contact structure. If we put

(1.8) $$2g_{ir}\varphi_j{}^r=2\varphi_{ji}=\partial_j\eta_i-\partial_i\eta_j,$$

then we can find four tensors $\varphi_j{}^i$, ξ^i, η_i and g_{ji} so that they define an (φ,ξ,η,g)-structure. Such a structure is called a contact metric structure [5].

In an almost contact metric space there are four tensor fields $N_{ji}{}^h$, $N_j{}^i$, N_{ji} and N_j which are the analogue of the Nijenhuis tensor in an almost complex structure [5].

A contact metric space with $N_{ji}=0$ or $N_{ji}{}^h=0$ is called a K-contact metric space or a normal contact metric space respectively. Of course a normal contact metric space is a K-contact metric space and a K-contact metric space is a contact metric space [6]. In the following we consider a notation η^i instead of ξ^i.

A K-contact metric space in which the Ricci tensor takes the form

(1.9) $$R_{ji}=ag_{ji}+b\eta_j\eta_i;$$

is called a K-contact η-Einstein space, where a and b become constant$(n>3)$, and

(1.10) $$a+b=n-1,\quad R=an+b$$

hold good [7], [6].

Let $R_{kji}{}^h$ be the Riemannian curvature tensor and put

(1.11) $$H_{ji}=\varphi^{kh}R_{kjih},\quad \text{then } H_{ji}=-\frac{1}{2}\varphi^{kh}R_{khji}.$$

In a contact metric space, φ_{ji} is a skew symmetric closed tensor and

(1.12) $$\nabla_r\varphi_j{}^r=(n-1)\eta_j$$

holds good, where ∇_i denotes the covariant differentiation with respect to the Riemannian connection.

In a K-contact metric space the following identities are valid [6]:

(1.13) $$\nabla_j\eta_i=\varphi_{ji},$$

(1.14) $$\nabla_k\varphi_{ji}+R_{rkji}\eta^r=0,$$

(1.15) $$R_{kjih}\eta^k\eta^j=0,\quad H_{ir}\eta^r=0,$$

(1.16) $$R_{kjih}\eta^k\eta^h=g_{ji}-\eta_j\eta_i,$$

(1.17) $$R_{ir}\eta^r=(n-1)\eta_i.$$

In a normal contact metric space

(1.18) $$\nabla_k\varphi_{ji}=\eta_j g_{ki}-\eta_i g_{kj},$$

(1.19) $$\eta_r R_{kji}{}^r=\eta_k g_{ji}-\eta_j g_{ki},$$

(1.20) $$\varphi_j{}^r R_{ri}=H_{ji}+(n-2)\varphi_{ji},$$

and also (1.13), (1.17) hold good [6].

In a normal contact or K-contat metric space a vector v^i is called an infinitesimal CL-transformation if it satisfies

(1.21) $$\underset{v}{\pounds}\left\{ {h \atop ji} \right\}=\rho_j\delta_i^h+\rho_i\delta_j^h+\alpha(\eta_j\varphi_i{}^h+\eta_i\varphi_j{}^h),$$

where $\underset{v}{\pounds}$ is the operator of Lie derivative and $\left\{ {h \atop ji} \right\}$ is Riemannian connection, ρ_i is a vector field and α is a certain scalar [1], [3]. Contracting h and j in (1.21), we see that ρ_i is a gradient.

In a K-contact metric space an infinitesimal CL-transformation hold good the following relations [4].

(1.22) $$\underset{v}{\pounds}R_{ji}=(1-n)\nabla_j\rho_i+2\alpha(n\eta_j\eta_i-g_{ji})+\eta_j\varphi_i{}^r\nabla_r\alpha+\eta_i\varphi_j{}^r\nabla_r\alpha,$$

(1.23) $$\eta_h\underset{v}{\pounds}R_{kji}{}^h=\eta_j\nabla_k\rho_i-\eta_k\nabla_j\rho_i+\alpha(\eta_k g_{ji}-\eta_j g_{ki}).$$

Finally we shall prepare the following theorem which has been proved by H. Mizusawa and K. Takamatsu.

LEMMA. *In a normal contact metric space, if v^i is an infinitesimal CL-transformation, then the following relation holds good* [3].

(1.24) $$\underset{v}{\pounds}g_{ji}=-\nabla_j\rho_i+\alpha(g_{ji}+\eta_j\eta_i).$$

§2. Infinitesimal CL-transformations in an η-Einstein normal contact metric space.

Let v^i be an infinitesimal CL-transformation in a normal contact metric space. Substituting (1.21) and (1.24) into the identity

$$\nabla_k\underset{v}{\pounds}g_{ji}=g_{hi}\underset{v}{\pounds}\left\{ {h \atop kj} \right\}+g_{jh}\underset{v}{\pounds}\left\{ {h \atop ki} \right\},$$

we get

(2.1) $$R_{rij}{}^h\rho^r+(g_{ji}+\eta_j\eta_i)\nabla^h\alpha-(\delta_i^h+\eta^h\eta_i)\nabla_j\alpha=\rho^h g_{ji}-\rho_j\delta_i^h,$$

$$\underset{\rho}{\pounds}\left\{ {h \atop kj} \right\}+2(\rho_k\delta_j^h+\rho_j\delta_k^h)=(\delta_k^h+\eta^h\eta_k)\nabla_j\alpha+(\delta_j^h+\eta^h\eta_j)\nabla_k\alpha-(g_{jk}+\eta_j\eta_k)\nabla^h\alpha \quad [4].$$

Thus we have the following:

PROPOSITION 2.1. *Let v^i be an infinitesimal CL-transformation and ρ_i be its associated vector. If α is constant then ρ_i is an infinitesimal projective transformation and conversely.*

Now, we begin with some simple lemmas.

LEMMA 2.2. *In a K-contact metric space, for a vector field ρ_i if there exist λ and μ such that*

$$\nabla_j\rho_i=\lambda g_{ji}+\mu\eta_j\eta_i, \tag{2.2}$$

then we have $\mu=0$ [7].

PROOF. Differentiating (2.2) covariantly and taking account of (1.13) we have

$$\nabla_k\nabla_j\rho_i=g_{ji}\nabla_k\lambda+\eta_j\eta_i\nabla_k\mu+\mu(\varphi_{kj}\eta_i+\varphi_{ki}\eta_j).$$

Transvecting φ^{kj} to this and making use of (1.4), (1.5) and (1.11), we get

$$H_{ir}\,\rho^r=-\varphi_i{}^k\nabla_k\lambda+\mu(n-1)\eta_i.$$

Transvecting the last equation with η_i and using of (1.15), we have $\mu=0$. This complets the proof.

LEMMA 2.3. *In a K-contact metric space, for a vector field ρ_i if there exist scalars λ and μ such that*

$$\nabla_j\rho_i=\lambda g_{ji}+\mu\eta_j\eta_i+c(\eta_j\varphi_i{}^r\rho_r+\eta_i\varphi_j{}^r\rho_r),\quad c=constant, \tag{2.3}$$

then we have $\mu=c\lambda$.

PROOF. Operating ∇_k to (2.3), using of (1.13), we get

$$\begin{aligned}\nabla_k\nabla_j\rho_i=\;&g_{ji}\nabla_k\lambda+\eta_j\eta_i\nabla_k\mu+\mu(\varphi_{kj}\eta_i+\varphi_{ki}\eta_j)+c\{\varphi_{kj}\varphi_{ir}\rho^r+\varphi_{ki}\varphi_{jr}\rho^r\\&+\eta_j\rho^r\nabla_k\varphi_{ir}+\eta_i\rho^r\nabla_k\varphi_{jr}+\eta_j\varphi_{ir}\nabla_k\rho^r+\eta_i\varphi_{jr}\nabla_k\rho^r\}.\end{aligned} \tag{2.4}$$

On the other hand, from (2.3) we have

$$\varphi_{ir}\nabla_k\rho^r=\lambda\varphi_{ik}-c\eta_k\rho_i+c\eta_r\rho^r\eta_k\eta_i.$$

Substituting (1.14) and the last equation into (2.4), we obtain

$$\begin{aligned}\nabla_k\nabla_j\rho_i=\;&g_{ji}\nabla_k\lambda+\eta_j\eta_i\nabla_k\mu+\mu(\varphi_{kj}\eta_i+\varphi_{ki}\eta_j)\\&+c\{\varphi_{kj}\varphi_{ir}\rho^r+\varphi_{ki}\varphi_{jr}\rho^r-\eta_j\rho_r\eta^s R_{ski}{}^r-\eta_i\rho_r\eta^s R_{skj}{}^r\end{aligned}$$

$$+\eta_i(\lambda\varphi_{jk}-c\ \eta_k\rho_j+c\ \eta^r\rho_r\eta_k\eta_j)+\eta_j(\lambda\varphi_{ik}-c\ \eta_k\rho_i+c\ \eta^r\rho_r\eta_k\eta_i)\}.$$

Transvecting $\varphi^{kj}\eta^i$ to this and making use of (1.4), (1.5), (1.16) and (1.15), we get $\mu=c\lambda$.

LEMMA 2.4. *In an n ($n>3$) dimensional normal contact η-Einstein space ($b\neq 0$), v^i is an infinitesimal CL-transformation then the following relation holds good.*

$$\text{(2.5)} \qquad \nabla_j\ \alpha=\frac{b}{n}(\rho_j-\eta_r\rho^r\eta_j).$$

PROOF. Contracting h and j in (2.1) we have

$$\text{(2.6)} \qquad -R_{rj}\rho^r+\eta_j\eta^r\nabla_r\alpha-n\nabla_j\alpha=(1-n)\rho_j.$$

Transvecting (2.6) with η^j and using of (1.17), we get $\eta^r\nabla_r\alpha=0$.
Thus (2.6) can be written as

$$R_{jr}\rho^r+n\nabla_j\alpha=(n-1)\rho_j.$$

Substituting (1.9) and (1.10) into the last equation, we obtain (2.5).

LEMMA 2.5. *Let v^i be an infinitesimal CL-transformation in a normal contact η-Einstein space ($n>3$) with $b\neq 0$, then v^i is a contact one.*

PROOF. Taking of the Lie derivative of the both sides of (1.19) and substituting (1.23) into the equation thus obtained, we get

$$\text{(2.7)} \qquad R_{kji}{}^h\underset{v}{\pounds}\eta_h=g_{ji}\underset{v}{\pounds}\eta_k+\eta_k\underset{v}{\pounds}g_{ji}-g_{ki}\underset{v}{\pounds}\eta_j-\eta_j\underset{v}{\pounds}g_{ki}-\eta_j\nabla_k\rho_i$$
$$+\eta_k\nabla_i\rho_j+\alpha(\eta_j g_{ki}-\eta_k g_{ji}).$$

Transvecting (2.7) with φ^{kj}, we have

$$\text{(2.8)} \qquad (\varphi^{kj}R_{kji}{}^h+2\varphi_i{}^h)\underset{v}{\pounds}\eta_h=0.$$

Substituting (1.9), (1.11) and (1.20) into (2.8), we get

$$\text{(2.9)} \qquad \underset{v}{\pounds}\eta_j=\sigma\ \eta_j\ ,$$

where we have put $\sigma=\eta^r\underset{v}{\pounds}\eta_r$.

In an η-Einstein space with $b\neq 0$, for any vector v^i we have

$$\text{(2.10)} \qquad \underset{v}{\pounds}R_{ji}=a\underset{v}{\pounds}g_{ji}+b(\eta_i\underset{v}{\pounds}\eta_j+\eta_j\underset{v}{\pounds}\eta_i).$$

Substituting (1.22), (1.24), (2.5) and (2.9) into (2.10), we obtain

(2.11) $$(1-n)\nabla_j\rho_i+2\alpha(n\eta_j\eta_i-g_{ji})+\frac{b}{n}(\eta_j\varphi_i{}^r\rho_r+\eta_i\varphi_j{}^r\rho_r)$$
$$=a[-\nabla_j\rho_i+\alpha(g_{ji}+\eta_j\eta_i)]+2\,b\,\sigma\eta_j\eta_i.$$

THEOREM 2.6. *In a normal contact η-Einstein space ($n>3$) with $a+2<0$, v^i be an infinitesimal CL-transformation with α=constant, then v^i is a concircular one.*

PROOF. From Lemma 2.4, (2.11) can be written as

(2.12) $$(1-n)\nabla_j\rho_i+2\alpha(n\eta_j\eta_i-g_{ji})=a[-\nabla_j\rho_i+\alpha(g_{ji}+\eta_j\eta_i)]+2\,b\,\sigma\eta_j\eta_i.$$

Applying Lemma 2.2 to (2.12), it follows that

$$-b\nabla_j\rho_i=(a+2)\alpha g_{ij},$$

which shows that the transformation is concircular.

THEOREM 2.7. (*) *In a compact normal contact η-Einstein space ($n>3$) with $a+2<0$, let v^i be an infinitesimal CL-transformation then v^i is an infinitesimal isometry.*

PROOF. Operating ∇_k to (2.5), we have

$$\nabla_k\nabla_j\alpha=\frac{b}{n}(\nabla_k\rho_j-\varphi_k{}^r\rho_r\eta_j-\eta_r\eta_j\nabla_k\rho^r-\eta^r\rho_r\varphi_{kj}).$$

Transvecting g^{kj} to this and using of (1.4) we get

(2.13) $$\Delta\alpha=\frac{b}{n}(\nabla^r\rho_r-\beta),$$

where we put $\beta=\eta^r\eta^s\nabla_r\rho_s$.

On the other hand, substituting (1.9), (1.10), (1.22) and (1.24) into the identity

$$\underset{v}{\mathfrak{L}}R=g^{ji}\underset{v}{\mathfrak{L}}R_{ji}+R_{ji}\underset{v}{\mathfrak{L}}g^{ji},$$

we obtain

$$0=(1-n)\nabla^r\rho_r+(ag_{ji}+b\eta_j\eta_i)[\nabla^j\rho^i-\alpha(g^{ji}+\eta^j\eta^i)]\quad\text{or}$$

(2.14) $$b(\nabla^r\rho_r-\beta)=-(a+2)(n-1)\alpha.$$

Comparing with (2.13) and (2.14), it follows that

(2.15) $$\Delta\alpha=-\frac{n-1}{n}(a+2)\alpha.$$

(*) It is well known that $v^i+\frac{1}{2}\rho^i$ is an infinitesimal isometry [3].

Since $a+2<0$, applying Green's theorem to (2.15), we have $\alpha=0$ [9].
Last, applying Lemma 2.3 to (2.11), we get

$$n\nabla_j\rho_i+\frac{(n-1)(a+2)}{b}\alpha g_{ji}-(\eta_j\varphi_i{}^r\rho_r+\eta_i\varphi_j{}^r\rho_r)=0. \tag{2.16}$$

Thus, taking account of (2.5) and $\alpha=0$, we have $\nabla_j\rho_i=0$.

Since our space is compact, we find $\rho_i=0$.
Hence v^i is an infinitesimal isometry.

In an η-Einstein space it is known that if $\underset{v}{\mathfrak{L}}g_{ji}=0$, then $\underset{v}{\mathfrak{L}}\eta_i=0$ holds good [7].

By Theorem 2.7 and the identity

$$\nabla_j\underset{v}{\mathfrak{L}}\eta_i-\underset{v}{\mathfrak{L}}\varphi_{ji}=\eta_r\underset{v}{\mathfrak{L}}\left\{{r \atop ji}\right\},$$

we have immediately the following [2]:

COROLLARY 2.8. *In a compact normal contact η-Einstein space ($n>3$) with $a+2<0$, an infinitesimal CL-transformation is an automorphism.*

§ 3. Curvature-preserving infinitesimal CL-transformation in a K-contact metric space.

M. Okumura has proved that, in a normal contact metric space any curvature-preserving infinitesimal transformation is necessary an infinitesimal isometry [8].

In this section we shall prove the following:

THEOREM 3.1. *In a compact K-contact metric space, a curvature preserving infinitesimal CL-transformation is necessary an infinitesimal isometry.*

PROOF. Transvecting g^{ji} to (1.22), we have $\nabla_r\rho^r=0$. Therefore $\rho_i=0$.

Transvecting (1.22) with $\eta^j\eta^i$, we get

$$(1-n)\eta^r\eta^s\nabla_r\rho_s+2\alpha(n-1)=0. \tag{3.1}$$

On the other hand, transvecting $g^{ki}\eta^j$ to (1.23) and taking account of $\nabla^r\rho_r=0$, we obtain

$$-\eta^r\eta^s\nabla_r\rho_s-\alpha(n-1)=0. \tag{3.2}$$

From (3.1) and (3.2), we find $\alpha=0$, and hence $\underset{v}{\mathfrak{L}}\left\{{h \atop ji}\right\}=0$.

Since our space is compact, we have $\underset{v}{\mathfrak{L}}g_{ji}=0$. This completes the proof.

In the proof of Theorem 3.1, we have immediately the following

COROLLARY 3.2. *Let v^i be an infinitesimal CL-transformation and ρ_i be its associated vector in a K-contact metric space. In order that v^i be an infinitesimal curvature-preserving transformation, it is necessary and sufficient that α be zero and $\nabla_j \rho_i=0$.*

§4. Infinitesimal η-conformal transformation.

In a contact metric space, we consider an infinitesimal transformation satisfying the following

$$(4.1) \qquad \underset{v}{\mathcal{L}} g_{ji}=\lambda(g_{ji}+\eta_j\eta_i),$$

where λ is a scalar function. We shall call such a transformation an infinitesimal η-conformal one [2]. In the paper [2], H. Mizusawa has proved the following two theorems.

THEOREM A. *In a K-contact metric space with constant scalar curvature $R \neq -(n-1)$, an infinitesimal η-conformal transformation with λ=constant is an infinitesimal isometry.*

THEOREM B. *In order that a transformation in a contact metric space be an infinitesimal isometry, it is necessary and sufficient that the transformation be infinitesimal η-conformal and infinitesimal affine at the same time.*

Now, we shall prove the following :

THEOREM. 4.1. *In a compact K-contact metric space ($n>3$) with constant scalar curvature $R+(n-1)\leqq 0$, an infinitesimal η-conformal transformation is an infinitesimal isometry.*

PROOF. Substituting (4.1) into the identity

$$\underset{v}{\mathcal{L}}\left\{ {h \atop ji} \right\}=\frac{1}{2}g^{hr}(\nabla_j \underset{v}{\mathcal{L}} g_{ri}+\nabla_i \underset{v}{\mathcal{L}} g_{rj}-\nabla_r \underset{v}{\mathcal{L}} g_{ji}),$$

we get

$$(4.2) \qquad \underset{v}{\mathcal{L}}\left\{ {h \atop ji} \right\}=\frac{1}{2}[\lambda_j(\delta_i^h+\eta^h\eta_i)+\lambda_i(\delta_j^h+\eta^h\eta_j)-\lambda^h(g_{ji}+\eta_j\eta_i)$$
$$+2\lambda(\varphi_j{}^h\eta_i+\varphi_i{}^h\eta_j)], \quad \lambda_i=\partial_i\lambda.$$

According to (1.14), (4.2) and the identity

$$(4.3) \qquad \underset{v}{\mathcal{L}} R_{kij}{}^h=\nabla_k \underset{v}{\mathcal{L}}\left\{ {h \atop ji} \right\}-\nabla_j \underset{v}{\mathcal{L}}\left\{ {h \atop ki} \right\},$$

we have

$$(4.4)\qquad \underset{v}{\pounds}R_{kji}{}^{h}=\frac{1}{2}[\eta_i(\lambda_k\varphi_j{}^h-\lambda_j\varphi_k{}^h)+\eta^h(\lambda_j\varphi_{ki}-\lambda_k\varphi_{ji}+2\lambda_i\varphi_{kj})$$
$$+\nabla_k\lambda_i(\delta_j^h+\eta^h\eta_j)-\nabla_j\lambda_i(\delta_k^h+\eta^h\eta_k)+\lambda_i(\varphi_k{}^h\eta_j-\varphi_j{}^h\eta_k)$$
$$-\nabla_k\lambda^h(g_{ji}+\eta_j\eta_i)+\nabla_j\lambda^h(g_{ki}+\eta_k\eta_i)-\lambda^h(2\eta_i\varphi_{kj}+\eta_j\varphi_{ki}$$
$$-\eta_k\varphi_{ji})+2\varphi_i{}^h(\lambda_k\eta_j-\lambda_j\eta_k)+2\lambda(2\varphi_{kj}\varphi_i{}^h+\varphi_{ki}\varphi_j{}^h-\varphi_{ji}\varphi_k{}^h)$$
$$+2\lambda(\eta_k\eta^r R_{rji}{}^h-\eta_j\eta^r R_{rki}{}^h-\eta^r\eta_i R_{jkr}{}^h)].$$

Now taking of the Lie derivative on both sides of (1.16), we obtain

$$(4.5)\qquad \eta^k\eta_h\underset{v}{\pounds}R_{kji}{}^h+\eta^k R_{kji}{}^h\underset{v}{\pounds}\eta_h+\eta_h R_{kji}{}^h\underset{v}{\pounds}\eta^k=\underset{v}{\pounds}g_{ji}-\eta_i\underset{v}{\pounds}\eta_j-\eta_j\underset{v}{\pounds}\eta_i.$$

Substituting (4.1) and (4.4) into (4.5), transvecting g^{ji} to (4.5), and making use of (1.15) and (1.16) we get

$$(4.6)\qquad \frac{1}{2}[4\beta-(n+1)\beta-2\nabla^r\lambda_r+2(n-1)\lambda]=\lambda(n+1)-2\eta^r\underset{v}{\pounds}\eta_r,$$

where we put $\beta=\eta^r\eta^s\nabla_r\lambda_s$.

On the other hand, from (4.1) and the identity

$$\frac{1}{2}\eta^r\eta^s\underset{v}{\pounds}g_{rs}=\eta^r\underset{v}{\pounds}\eta_r,$$

we have

$$(4.7)\qquad \eta^r\underset{v}{\pounds}\eta_r=\lambda\ .$$

Making use of (4.6) and (4.7), we obtain

$$(4.8)\qquad 2\nabla^r\lambda_r+(n-3)\beta=0.$$

According (4.1), (4.4) and the identity

$$g^{ji}\underset{v}{\pounds}R_{ji}+R_{ji}\underset{v}{\pounds}g^{ji}=\underset{v}{\pounds}R=0,$$

we have

$$(4.9)\qquad -n\nabla^r\lambda_r+\beta-\lambda(R+n-1)=0.$$

Substituting (4.9) into (4.8) to eliminate β, we get

$$(4.10)\qquad \nabla^r\lambda_r=-\frac{(n-3)(R+n-1)}{(n-1)(n-2)}\lambda,\quad (n>3).$$

Applying Green's theorem to (4.10), we have $\lambda=0$ if $R+n-1\leqq 0$. This completes the proof.

We have also from theorem 4.1.

COROLLARY 4.2. *In a compact K-contact η-Einstein space $(n>3)$ with a $+2<0$,* *an infinitesimal η-conformal transformation is an automorphism.*

THEOREM 4.3. *In order that a transformation in an Einstein (or compact) contact metric space be an infinitesimal isometry, it is necessary and sufficient that the transformation be infinitesimal η-conformal and infinitesimal CL-transformation at the same time.*

PROOF. By theorem B, the necessity is evident. We shall prove the sufficiency. From (1.21) and (4.2), it follows that

$$\rho_j\delta_i^{\ h}+\rho_i\delta_j^{\ h}+\alpha(\eta_j\varphi_i^{\ h}+\eta_i\varphi_j^{\ h}) \tag{4.11}$$

$$=\frac{1}{2}\left[\lambda_i(\delta_i^{\ h}+\eta^h\eta_i)+\lambda_i(\delta_j^{\ h}+\eta^h\eta_j)-\lambda^h(g_{ji}+\eta_j\eta_i)+2\lambda(\varphi_j^{\ h}\eta_i+\varphi_i^{\ h}\eta_j)\right].$$

Contracting (4.11) with respect to j and h, we get $2\rho_i=\lambda_i$.
Next transvecting (4.11) with η_h, we obtain

$$\eta_j\rho_i+\eta_i\rho_j=(\eta_r\rho^r)(g_{ji}+\eta_j\eta_i),$$

from which we have $\eta_r\rho^r=0$ and $\rho_i=0$. From these and (4.11), we find $\lambda=$constant, $\alpha=\lambda$.
By the identity (4.3) and $\lambda=$constant it follows that

$$\underset{v}{\pounds}R_{ji}=\lambda\nabla_r(\varphi_j^{\ r}\eta_i+\varphi_i^{\ r}\eta_j), \tag{4.12}$$

$$\underset{v}{\pounds}R=g^{ji}\underset{v}{\pounds}R_{ji}+R_{ji}\underset{v}{\pounds}g^{ji}=-\lambda(R+R_{ji}\eta^j\eta^i). \tag{4.13}$$

If we assume that the space be Einstein, we have

$$\underset{v}{\pounds}R=-\frac{n+1}{n}\lambda R=0,$$

from which we get $\lambda=0$. If the space be compact, from (4.1) we have

$$\nabla^r v_r=\frac{n+1}{2}\lambda\,.$$

By Green's theorem we have $\lambda=0$. These complete the proof.

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