A NOTE ON THE RIEMANN MAPPING THEOREM

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The Riemann Mapping Theorem [1] states that if R is a simply connected region which is not the whole complex plane C and a is a point of R, then there exists a unique analytic homeomorphism f_a of R onto the open unit disc $D=\{z \in C \mid |z|<1\}$ such that $f_a(a)=0$ and $f_a'(a)>0$. Let $H(R)=\{f_a \mid a \in R\}$ with the compact open topology. We prove the following.

THEOREM. *R and $H(R)$ are homeomorphic.*

PROOF. We define a function $F: R \to H(R)$ as follows: For each point $a \in R$, let $F(a)=f_a$, where f_a is provided by the Riemann Mapping Theorem. Then obviously F is one-to-one and onto. To establish the continuity of F, we let a_1, a_2, $a_3 \cdots\cdots$ be a sequence of points of R which converges to a point $a_0 \in R$, and will show that the sequence $F(a_1)$, $F(a_2)$, $F(a_3), \cdots$ converges to $F(a_0)$ in $H(R)$. To avoid the use of double subscripts, we let $F(a_n)=f_n$. Denote the points $f_0(a_n)$ by b_n, $n=0,1,2,\cdots\cdots$. Then by the continuity of f_0, the sequence b_1, b_2, $b_3, \cdots$ converges to b_0, where $b_0=f_0(a_0)$. Since the sequence $f_0'(a_1)$, $f_0'(a_2)$, $f_0'(a_3), \cdots$ converges to $f_0'(a_0)$, where $f_0'(a_0)>0$, there exists a positive integer N such that $|f_0'(a_n)|>0$ for all $n \geqq N$. Therefore, for each integer $n \geqq N$, there exists a complex number k_n such that $|k_n|=1$, $k_n f_0'(a_n)>0$, and the sequence $\{k_n\}_n$ converges to 1. Then, for each $n \geqq N$, $k_n f_0$ is an analytic homeomorphism of R onto D, and the sequence $\{k_n f_0\}_n$ converges to $k_0 f_0$ uniformly on every compact subset of R. Now for each $n \geqq N$, let $c_n=k_n b_n$ and g_n be the unique analytic homeomorphism of D onto itself such that $g_n(c_n)=0$ and $g_n'(c_n)>0$. Let $h_n=g_n(k_n f_0)$, for each $n \geqq N$. Then each h_n is an analytic homeomorphism of R onto D such that $h_n(a_n)=0$ and $h_n'(a_n)>0$. Thus by the uniqueness of the Riemann Mapping Theorem, we must conclude that $h_n=f_n$ for each $n \geqq N$.

On the other hand, since each g_n is an analytic function of D onto itself, each g_n can be written as $g_n(z)=(\alpha_n z+\bar{\beta}_n)/(\beta_n z+\bar{\alpha}_n)$ with the condition that $\alpha_n\bar{\alpha}_n-\beta_n\bar{\beta}_n=1$. But, solving for α_n and β_n in terms of c_n, we obtain either $\alpha_n=(1-c_n\bar{c}_n)^{-1/2}$ and $\beta_n=-\bar{c}_n/(1-c_n\bar{c}_n)^{1/2}$ or $\alpha_n=-(1-c_n\bar{c}_n)^{-1/2}$ and $\beta_n=\bar{c}_n/(1-c_n\bar{c}_n)^{1/2}$. Now, it follows that the sequence $\{g_n\}_n$ converges uniformly to the identity function g_0 on every compact subset of D. This fact together with the uniform convergence of the sequence $\{k_n f_0\}_n$ on every compact subset of R implies that the sequence $\{h_n\}_n$ converges to h_0 uniformly on every compact subset of R. Hence the sequence $F(a_1)$, $F(a_2)$, $F(a_3)$, …… converges to $F(a_0)$ in $H(R)$.

Since the topology for $H(R)$ is metrizable and separable, to establish the continuity of F^{-1}, it is sufficient to show the sequential continuity of F^{-1}. Let f_1, f_2, f_3, …… be a sequence of elements of $H(R)$ which converges to $f_0 \in H(R)$. Recall that the convergence of a sequence in $H(R)$ is uniform convergence on every compact subset of R. Let $a_0=F^{-1}(f_0)$. We may now apply Hurwitz's Theorem [1] to conclude that every sufficiently small neighborhood U of a_0 contains exactly one zero of each f_n if n is sufficiently large. If we denote $a_n=F^{-1}(f_n)$, $n=1,2,3,$……, each a_n is a zero of f_n, and by the Riemann Mapping Theorem the only zero. Hence, for some N, if $n\geqq N$, $a_n \in U$. Therefore, a_0 is a limit point of the sequence a_1, a_2, a_3, ……. It is obvious that a_0 is the only limit point of the sequence. Hence, the continuity of F^{-1} is proved.

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BIBLIOGRAPHY

[1] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, 1953.
[2] J. Dugundji, *Topology*, Allyn and Bacon, 1966