ON PRECORRECT UNIFORM SPACES

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A subset $\mathscr{U}$ of the power set of $(X\times X)$ is a *precorrect uniformity* on X iff $\mathscr{U}$ satisfies (A_1) $U\in\mathscr{U}$ iff $U=U^{-1}$, $U\supset\Delta$, and U contains a member of $\mathscr{U}$; (A_2) For every $A\subset X$ and U,V in $\mathscr{U}$ there exists a $W\in\mathscr{U}$ such that $W[A]\subset U[A]\cap V[A]$; (A_3) For every $A\subset X$ and $U\in\mathscr{U}$ there exists V, W in $\mathscr{U}$ such that $(W\circ V)[A]\subset U[A]$.

A relation δ on $P(X)$, the power set of X, is a *proximity* on X iff δ satisfies: (P_1) $A\delta B$ implies $B\delta A$; (P_2) $C\delta(A\cup B)$ iff either δCA or $C\delta B$; (P_3) $\phi\bar{\delta}A$ for every $A\subset X$; (P_4) $x\delta x$ for all $x\in X$; (P_5) $A\bar{\delta}B$ implies the existence of C and D such that $C\cap D=\phi$, and $A\bar{\delta}(X\text{-}D)$, $B\bar{\delta}(X-D)$

THEOREM 1. *Let $\mathscr{U}$ be a subset of the power set of $(X\times X)$. Suppose for each $U\in\mathscr{U}$ $U=U^{-1}$. Define a relation $\delta(\mathscr{U})$ on $P(X)$ by $A\delta(\mathscr{U})B$ iff $U[A]\cap B\neq\phi$ for all $U\in\mathscr{U}$. Then $\delta(\mathscr{U})$ satisfies (P_1), (P_2), (P_3), (P_4) and (P_5) iff $\mathscr{U}$ satisfies $(A_1{}^*)$: $U\in\mathscr{U}$ implies $U\supset\Delta$, (A_2) and (A_3).*

A proof of Theorem 1 is given in [1].

If we are given δ, a proximity on X, then the class of precorrect uniformities $\mathscr{U}$ on X such that $\delta(\mathscr{U})=\delta$ is called a *proximity class of precorrect uniformities* on X and is denoted by $\Pi(\delta)$.

THEOREM 2. *Let (X,δ) be a proximity space. Then $\Pi(\delta)$ contains one and only one totally bounded symmetric uniformity.*

PROOF. This is an immediate consequence of Theorem 21.20 in [6].

THEOREM 3. *Let (X,δ) be a proximity space Then $\Pi(\delta)$ contains a maximum and a minimum.*

PROOF. For all A, B in P(X) let $U_{A,B}=(X\times X)-((A\times B)\cup(B\times A))$. It is easy to show by Theorem 1 that $\mathscr{B}=\{U_{A,B}|A\ \bar{\delta}B\}$ is a base for a precorrect uniformity $\mathscr{U}_1(\delta)$ on X such that $\mathscr{U}_1(\delta)$ is the least element in $\Pi(\delta)$. Also, it is easily shown by Theorem 1 that the union of an arbitrary family of members of $\Pi(\delta)$ is a base for a precorrect uniformity on X that is a member of $\Pi(\delta)$; consequently $\Pi(\delta)$ has a maximum element.

THEOREM 4. *If δ is the usual proximity for the reals, X, then $\Pi(\delta)$ contains at least two distinct precompact precorrect uniformities that have an open base.*

PROOF. Let $\gamma=\{U_{A,B} \mid A\bar{\delta}B\}$. Let $\mathscr{B}=\{$all finite intersections of members of $\gamma\}$. It can be shown by Theorem 1 that $\mathscr{B}$ is a base for a precompact symmetric uniformity $\mathscr{U}_2(\delta)$ on X such that if $\mathscr{U}_1(\delta)$ is the uniformity that was constructed in Theorem 3 then $\mathscr{U}_1(\delta)$ is properly contained in $\mathscr{U}_2(\delta)$. It is easily shown that both $\mathscr{U}_1(\delta)$ and $\mathscr{U}_2(\delta)$ are totally bounded and have an open base (cf. [5]).

We say that a filter in the precorrect uniform space $(X, \mathscr{U})$ is *weakly Cauchy* iff for every $U \in \mathscr{U}$ there exists an $x \in X$ such that $U[x] \in \mathscr{F}$. Also, $(X, \mathscr{U})$ is *complete* iff every weakly Cauchy filter on $(X, \mathscr{U})$ has a cluster point in X.

THEOREM 5. *If $(X, \mathscr{T})$ is a connected completely regular topological space, then there exists a precompact precorrect uniformity $\mathscr{U}$ on X with an open base such that $\mathscr{T}(\mathscr{U})=\mathscr{T}$ and every filter is weakly Cauchy.*

PROOF. Let δ be a proximity on X such that $\mathscr{T}(\delta)=\mathscr{T}$. Let $\mathscr{U}_1(\delta)$ be an element of $\Pi(\delta)$. Note that $\mathscr{U}_1(\delta)$ exists by Theorem 3. Let $U \in \mathscr{U}_1(\delta)$. Then there exist sets $A \subset X$ and $B \subset X$ such that $U \supset U_{A,B} \supset U_{\bar{A},\bar{B}}$. But since $\mathscr{T}$ is connected, there exists $x_0 \in (X-(\bar{A} \cup \bar{B}))$: so that $U_{\bar{A},\bar{B}}[x_0]=X$. Hence every filter on X is weakly Cauchy with respect to $\mathscr{U}$.

THEOREM 6. *A precorrect space is compact iff it is complete and precompact.*

PROOF. This is an immediate consequence of Theorem 4 in [4] and the easily established fact namely that every precorrect uniform space is a symmetric generalized uniform space (cf. [1]).

THEOREM 7. *A completely regular topological space is compact iff it is complete with respect to every compatible precorrect uniformity on X.*

PROOF. This is an immediate consequence of the Lemma on page 5 in [3] and Theorem 6 above.

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REFERENCES

[1] A. G. Mordkovič, *Test for the correctness of a uniform space*. Soviet Math. Dokl., 7 (1966), pp. 915—917.

[2] C. J. Mozzochi, *A partial generalization of a theorem of Hursch*, CMUC Commentationes Math. Univ. Carolinae, 9 (1968) pp. 157—159.

[3] C. J. Mozzochi, *A correct system of axioms for a symmetric generalized uniform space* (to appear) Math. Scand.

[4] C. J. Mozzochi, *On the completeness of a symmetric generalized uniform space* (pending) Math. Scand.

[5] C. J. Mozzochi, *A generalization of a theorem of Alfsen and Fenstad* (pending) Math. Scand.

[6] W. J. Thron, *Topological Structures*, Holt, Rinehart, and Winston, New York, 1966.