THE STRENGTHENING OF TOPOLOGIES BETWEEN T_0 AND T_1

By Young Sik Park.

1. Introduction.

Separation axioms was studied by several mathematicians in early days. The only separation axioms between T_0 and T_1 known heretofore was introduced by J. W. T. Youngs. A number of new separation axioms between T_0 and T_1 was introduced by C. E. Aull and W. J. Thron. After introducing some definitions already known which will play an important role through this article, we use these to give a proof of question whether the separation axioms between T_0 and T_1 are preserved under strengthening or not. It is known that T_0, T_1, T_D, and T_{DD} are preserved under a strengthening of the topology. Moreover, T_2 satisfies the requirement, but T_3 and T_4 do not. The axiom $T(\partial)$ does not satisfy this condition. We employ the terminology and notation used by Kelley [2].

A set X is said to be weakly separated from Y (notation: $X \vdash Y$) iff there exists an open set $V \supset X$ such that $V \cap Y = \phi$. Instead of $[x] \vdash [y]$, we shall simply write $x \vdash y$ and say that x can be weakly separated from y. The closure of a point x (or more precisely of the set $[x]$), which will be denoted by $[\bar{x}]$, consists of those and only those points of the space for which $y \not\vdash x$. The derived set of a point, denoted by $[x]'$, consists of all $y \neq x$ for which $y \not\vdash x$. A degenerate set we shall mean a set which consists at most one point.

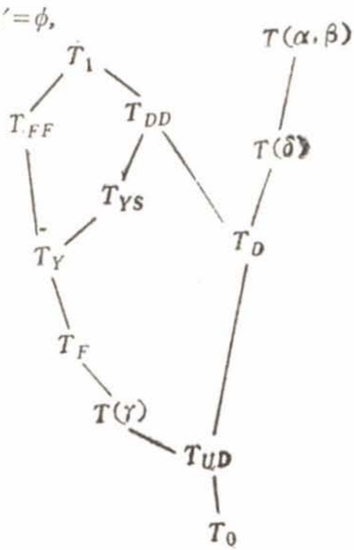
In the following, notation $(X, \mathscr{T})$ denotes arbitrary topological space. A space $(X, \mathscr{T})$ will be called a T_{UD}-space iff, for every $x \in X$, $[x]'$ is the union of disjoint closed sets. The space will be called a T_D-space iff, for every $x \in$X, $[x]'$ is a closed set. The space will be called a T_{DD}-space iff it is a T_D-space, and in addition, for all x, $y \in X$, $x \neq y$, $[x]' \cap [y[' = \phi$. A space $(X, \mathscr{T})$ will be called a T_F-space iff, given any point x and any finite set F in X, such that $x \notin F$, then either $x \vdash F$ or $F \vdash x$. A space will be called a T_{FF}-space iff, given two arbitrary finite sets F_1 and F_2 in X, with $F_1 \cap F_2 = \phi$, then either $F_1 \vdash F_2$ or $F_2 \vdash F_1$. A space $(X, \mathscr{T})$ is called a T_Y-space iff, for all $x, y \in X$, $x \neq y$, $[\bar{x}] \cap [\bar{y}[$ is degenerate. A space is called a T_{YS}-space iff, for all $x, y \in X$, $x \neq y$,

$[\bar{x}] \cap [\bar{y}]$ is either ϕ or $[x]$ or $[y]$. A space $(X, \mathscr{T})$ is called a $T(\gamma)$-space iff, for all $x \in X$, $[x]'$ is union of disjoint point closures.

2. The ordering relation.

The following chart shows the ordering relation between seperation axioms introduced by C. E. Aull and W. J. Thron. In this chart, α, β and δ represents the following axioms:

α: for every $x \in X$, $[x]'$ consists of points such $[y]' = \phi$,
β: for every $x \in X$, $[x]'$ is degenerate,
δ: for every $x \in X$, $[x]'$ is a point closure.



3. Behavior of C. E. Aull and W. J. Thron's separation axioms.

THEOREM 3.1. *If a space $(X, \mathscr{T})$ is a T_F-space, and $\mathscr{T}_1$ is a strengthening of the topology $\mathscr{T}$, then the space $(X, \mathscr{T}_1)$ is also a T_F-space.*

PROOF. Let $[\bar{x}]$ be the closure of a point x (or more precisely of the set $[x]$ in topology $\mathscr{T}_1$), and let $[\bar{x}]_\circ$ be the closure of a point x in topology $\mathscr{T}$. Let $[y]'$ be the derived set of a point y in topology $\mathscr{T}_1$ and let $[y]_\circ'$ be the derived set of a point y in topology $\mathscr{T}$. Then it is evident that the closure of a point x in topology $\mathscr{T}$ includes the closure of a point x in topology $\mathscr{T}_1$. That is, $[\bar{x}] \subseteq [\bar{x}]_\circ$. If $y \in [x]'$ then $y \in [\bar{x}]_\circ'$. Hence $y \in [x]_\circ'$ and $[y]_\circ' = \phi$. Therefore $[y]'$ must be empty, since $[y]'$ is included in $[y]_\circ'$.

THEOREM 3.2. A T_Y-*space is preserved under a strengthening of the topology.*

PROOF. Let a space $(X, \mathscr{T})$ be a T_Y-space and $\mathscr{T}_1$ is a strengthening of the topology $\mathscr{T}$. Then we know that the closure of a pint x in $\mathscr{T}_1$ is included in the closure of a point x in $\mathscr{T}$. Since $[\bar{x}]_\circ \cap [\bar{y}]_\circ$ is degenerate, $[\bar{x}] \cap [\bar{y}]$ must be degenerate.

THEOREM 3.3. *If a space $(X, \mathscr{T})$ is a T_{YS}-space and $\mathscr{T}_1$ is a strengthening of the topclogy $\mathscr{T}$, then $(X, \mathscr{T}_1)$ is also a T_{YS}-space.*

PROOF. It is proved by the same method as theorem 3.2.

THEOREM 3.4. *If a space $(X, \mathscr{T})$ is a T_{FF}-space and $\mathscr{T}_1$ is a strengthening of the topology $\mathscr{T}$, then the space $(X, \mathscr{T}_1)$ is a T_{FF}-space.*

PROOF. By the definition of T_{FF}, $[x]_\circ'$ is empty for all but at most one $x \in X$. Hence $[x]'$ has also this property.

THEOREM 3.5. *If a space $(X, \mathscr{T})$ is a T_{UD}-space* and *$\mathscr{T}_1$ is a strengthening of the topology $\mathscr{T}$, then $(X, \mathscr{T}_1)$* is *a T_{UD}-space.*

PROOF. Let $\mathscr{T}_1 = \mathscr{T} \cup \{A_\alpha\}$ where A_α is closed set for arbitrary α contained in index set. By the definition of T_{UD}-space, $[x]_\circ'$ is constructed by a union of some disjoint closed sets. We will prove that $[x]'$ is the union of disjoint closed sets for every $x \in (X, \mathscr{T}_1)$. If $x \notin \cup A_\alpha$, then it is evident. If $x \in \cup A_\alpha$ then we can consider two cases, If $[\bar{x}] \neq A \in \{A_\alpha\}$ then it is trivial. We suppose $[\bar{x}] = A \in \{A_\alpha\}$, then $A \subseteq [\bar{x}]_\circ$. We can construct the following inclusion relation :

$[x]' \subseteq A \cap [x]_\circ' = A \cap (\cup C_\alpha) = \cup (C_\alpha \cap A) = [x]'$, where $C_\alpha \cap A$ is disioint closed sets in $\mathscr{T}_1$, Hence $(X, \mathscr{T}_1)$ is a T_{UD}-space.

THEOREM 3.6. *A $T(\gamma)$-space is not preserved under a strengthening of the topology.*

PROOF. We can show this by the following counter example :
Let X be the set of natural numbers and 0, ∞. Let the closed sets of $(X, \mathscr{T})$ be all sets of the form $\{n, n+1, \cdots\cdots, \infty\}$, where $n = 0, 1, 2, 3, \cdots\cdots, \infty$.

Then the topological space $(X, \mathscr{T})$ is $T(\gamma)$-space. Now adding of new closed sets, we construct the sets of the form $\{n, n+2, \cdots\cdots, \infty\}$: $n=0, 2, 4, 6, \cdots\cdots, \infty$. In the new space the derived set of $2n$ is not a union of disjoint point closure.

Department of Mathematics
Kyungpook University
Taegu, Korea.

REFERENCES

[1] Aull, C. E. and Thron, W. J.; *Separation axioms between T_0 and T_1*, Indagationes Math. Vol. 24, 26—37 (1962).

[2] Kelley, J. L.; *General Topology*. D. Van Nostrand (1955).