A NOTE ON EXTENSIONS OF TOPOLOGIES

By Duk Su Oh

In [1], L. Levine introduced the concept of the simple extension and investigated some topological properties which is preserved by the simple extension. Some results of L. Levine was extended by Carlos J. R. Borges [2], that is, the necessary and sufficient conditions for the simple extension of topology $\mathcal{T}$ to inherit complete regularity, hereditary normality, perfect normality, hereditary paracompactness, stratifiability, normality, paracompactness, Lindelöf, compactness and countable compactness from $\mathcal{T}$.

The purpose of this paper is to determine the followings;

(i) the necessary and sufficient condition for finite or infinite extension to be a simple extension.

(ii) the relation between the simple extension of product space and the product of simple extensions in the case of finite product.

(iii) the quasi-pseudo-metrizability of extension of topology $\mathcal{T}$.

DEFINITION. For a given topology $\mathcal{T}$ on a set X, the topology $\{U\cup(V\cap A) | U, V\in\mathcal{T}\}$, where A is a subset of X, is called a *simple extension* of $\mathcal{T}$ and denoted by $\mathcal{T}(A)$. [2]

It follows that $\mathcal{T}(A)$ is the topology with $\mathcal{T}\cup\{U\cap A | U\in\mathcal{T}\}$ as subbase.

Let $\mathcal{T}$ be a topology on X and $\mathcal{A}=\{A_\alpha | \alpha\in\Delta\}$ be a family of subsets of X, and let $\mathcal{T}_{A_\alpha}$ denotes the topology $\{A_\alpha\cap U | U\in\mathcal{T}\}\cup\{X\}$ for each $A_\alpha\in\mathcal{A}$. Then there exists a topology on X with $\mathcal{T}\cup\bigcup_{\alpha\in\Delta}\mathcal{T}_{A_\alpha}$ as subbase. We define this the *extension* of $\mathcal{T}$ by $\mathcal{A}$ and denote this $\mathcal{T}(\mathcal{A})$. $\mathcal{T}(\mathcal{A})$ is called *finite extension* or *infinite extension* according as Δ is finite or infinite. It follows that $\mathcal{T}(\mathcal{A})=\mathcal{T}(A_1)(A_2)\cdots(A_n)$ if $\Delta=\{1, 2, \cdots, n\}$.

For a subset M of X, we shall denote the complement $X-M$ by M' and the closure of M relative to the topology $\mathscr{T}$ by $\overline{M}$.

THEOREM 1. *Let* $\mathfrak{A}=\{A_\alpha|\alpha\in\Delta\}$ *be a pairwise disjoint family of subsets of a topological space* $(X,\mathscr{T})$ *and let* $A=\cup_{\alpha\in\Delta}A_\alpha$. *Then* $\mathscr{T}(\mathfrak{A})=\mathscr{T}(A)$ *iff* $A_\alpha\cap\overline{A-A_\alpha}=\theta$ *for each* $\alpha\in\Delta$.

PROOF. Suppose $A_\alpha\cap\overline{A-A_\alpha}=\theta$ for each $\alpha\in\Delta$. Clearly $\mathscr{T}(\mathfrak{A})\supset\mathscr{T}(A)$. To prove $\mathscr{T}(\mathfrak{A})\subset\mathscr{T}(A)$, it is sufficient to show $U\cap A_\alpha\in\mathscr{T}(A)$ for each $U\in\mathscr{T}$ and each $\alpha\in\Delta$. If $x\in U\cap A_\alpha$, there exists an open neighborhood O_x of x relative to the topology $\mathscr{T}$ such that $O_x\cap A_\alpha\subset U\cap A_\alpha$ and $O_x\cap(A-A_\alpha)=\theta$. Let $W_\alpha=\cup\{O_x|x\in U\cap A_\alpha\}$, then $W_\alpha\cap A=W_\alpha\cap A_\alpha=U\cap A_\alpha$, and hence $U\cap A_\alpha\in\mathscr{T}(A)$

Conversely, if $\mathscr{T}(\mathfrak{A})=\mathscr{T}(A)$, then $A_\alpha\in\mathscr{T}(A)$ for all $\alpha\in\Delta$, and A_α is represented by $A_\alpha=U\cup(V\cap A)$ for some $U,V\in\mathscr{T}$. From the fact $\mathfrak{A}$ is pairwise disjoint, it follows that $V\cap(A-A_\alpha)=\theta$, and hence $(A-A_\alpha)\cap(U\cup V)=\theta$. It follows that $(U\cup V)'\supset\overline{A-A_\alpha}$, and consequently $A_\alpha\cap\overline{A-A_\alpha}=\theta$, since $A_\alpha\subset U\cup V$.

COROLLARY 2. *If* $\mathfrak{A}=\{A_\alpha|\alpha\in\Delta\}$ *is a discrete family of a topological space* $(X,\mathscr{T})$, *then* $\mathscr{T}(\mathfrak{A})=\mathscr{T}(A)$, *where* $A=\cup_{\alpha\in\Delta}A_\alpha$.

PROOF. This is an immediate consequence of above theorem.

The following ways are also possible. For each $U\in\mathscr{T}$ and each $\alpha\in\Delta$, we have $U\cap A_\alpha=U\cap A_\alpha\cap(\cup_{\beta\neq\alpha}\overline{A}_\beta)'=A\cap U\cap(\cup_{\beta\neq\alpha}\overline{A}_\beta)'$. Since $\mathfrak{A}$ is a discrete family, it holds $\cup_{\beta\neq\alpha}\overline{A}_\beta=\overline{\cup_{\beta\neq\alpha}A_\beta}$, and consequently $U\cap A_\alpha\in\mathscr{T}(A)$.

COROLLARY 3. *Let* $\mathfrak{A}=\{A_n|n\in N\}$ *be a pairwise disjoint family of subsets of a topological space* $(X,\mathscr{T})$, *where* $N=\{1,2,\cdots,n\}$, $A=\cup_{n\in N}A_n$. *Then* $\mathscr{T}(A_{i_1})(A_{i_2})\cdots(A_{i_n})=\mathscr{T}(A)$ *for all permulation* $(i_1,i_2,\cdots,i_n)$ *of* $(1,2,\cdots,n)$, *iff* $\mathfrak{A}$ *is pairwise seperated.*

THEOREM 4. *Let* $(X_i,\mathscr{T}_i)$ *be a topological space for* $i=1,2,\cdots,n$, *and let* $\mathscr{T}(A_i)$ *be a simple extension of* $\mathscr{T}_i$, *and* $\mathscr{T}$, $\mathscr{T}^*$ *denote the product topologies*

$\prod_{i=1}^{n}\{(X_i, \mathcal{T}_i)\}$, $\prod_{i=1}^{n}\{(X_i, \mathcal{T}_i(A_i))\}$ *respectively.*

Then(i) $\mathcal{T}(\prod_{i=1}^{n} A_i) \subset \mathcal{T}^*$

(ii) $\mathcal{T}(\prod_{i=1}^{n} A_i) = \mathcal{T}^*$ *iff* $A_i \in \mathcal{T}_i$ *for each* i, *whenever* $A_i \neq X_i$.

PROOF (i) Straightforward, since $\prod_{i=1}^{n} A_i \in \mathcal{T}^*$.

(ii) Suppose that $A_i \bar{\in} \mathcal{T}_i$ for some i. We will point out that $X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_n$ is a member of $\mathcal{T}^*$, but not a member of $\mathcal{T}(\prod_{i=1}^{n} A_i)$. Let $X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_n = O_\alpha \cup (O_\beta \cap \prod_{i=1}^{n} A_i)$ for some O_α, $O_\beta \in \mathcal{T}$ and let us take $a = (a_1, \cdots, a_i, \cdots, a_n) \in X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_n$ such that $a_i \in A_i - A_i^0$ (*), $a_j \in X_j - A_j$ for $j \neq i$. Then, since $a \bar{\in} \prod_{i=1}^{n} A_i$, it follows that $a \in O_\alpha$. Hence $a_i \in P_i[O_\alpha] \subset A_i$, where P_i is the projection of $\prod_{i=1}^{n} X_i$ to X_i. Recalling that the projection of a product space to each of its coordinate space is open, it follows that $a_i \in A_i^0$, which is contradiction. The converse is clear.

If $(X, \mathcal{T})$ satisfies the second axiom of countability, $(X, \mathcal{T}(A))$ also does, and $(X, \mathcal{T})$ is quasi-pseudo-metrizable [3]. Hence we have

THEOREM 5. *If* $(X, \mathcal{T})$ *satisfies the second axiom of countability,* $(X, \mathcal{T}(\mathfrak{A}))$ *is quasi-pseudo-metrizable, where* $\mathfrak{A}$ *is a countable family of subsets of* X.

LEMMA 6. *If there is a base* $\mathcal{L}$ *of* $(X, \mathcal{T})$ *satisfying* (a), $(X, \mathcal{T})$ *is quasi-pseudo metrizable.*

(a): *for each* $x \in X$, *there exists a member* B_x *of* $\mathcal{L}_x = \{B \mid x \in B \in \mathcal{L}\}$ *such that* $B_x \subset B$ *for each* $B \in \mathcal{L}_x$. (*i. e.* B_x *is the first element of* $\mathcal{L}_x$ *with the ordering* $\subset$.)

PROOF. Setting

$$p(x, y) = \begin{cases} 1 : & \text{there exist infinitely many members of } \mathcal{L}_x, \text{ each of which does not contain } y, \\ \sum_{n=1}^{N} \frac{1}{2^n} : & \text{there exist only } N \text{ members of } \mathcal{L}_x, \text{ each of which does not contain } y, \\ 0 : & y \text{ belongs to every member of } \mathcal{L}_x, \end{cases}$$

(*) A_i^0 is the interior of A_i relative to the topology $\mathcal{T}_i$

it follows that p is a quasi-pseudo-metric on X. It is easily established that the topology determined by p equals to $\mathscr{T}$.

THEOREM 7. *If there is a base* $\mathscr{L}$ *of* $(X, \mathscr{T})$ *satisfying (a), and* $\mathcal{O}\!\mathcal{L}=\{A_\alpha | \alpha\in\Delta\}$ *is a pairwise disjoint family of subsets of* X *such that* $A_\alpha \cap \overline{A-A_\alpha}=\emptyset$ *for each* $\alpha\in\Delta$, *where* $A=\bigcup_{\alpha\in\Delta}A_\alpha$, *then* $\mathscr{T}(\mathcal{O}\!\mathcal{L})$ *is quasi-pseudo metrizable.*

PROOF. By theorem 1 it follows that $\mathscr{L}^* =\mathscr{L}\cup\{B\cap A | B\in\mathscr{L}\}$ is a base of $\mathscr{T}(\mathcal{O}\!\mathcal{L})$. Let B_x be the first element of $\mathscr{L}_x$ with the ordering $\subset$, then $B_x\cap A$ or B_x is the first element of $\mathscr{L}_x{}^*$ according as x belongs to A or not. Therefore $\mathscr{T}(\mathcal{O}\!\mathcal{L})$ is quasi-pseudo-metrizable by lemma 6.

Department of Mathematics
Kyungpook University
Taegu, Korea.

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