A NOTE ON QUASI-UNIFORM SPACES

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1. Introduction

In [1], A, Császár introduced quasi-uniform spaces and investigated some properties of it. J. L. Sieber and W. J. Pervin [3], and R.C. Stoltenberg [4] made contributions to the theory of quasi-uniform completion. The purpose of the present note is to determine the followings ;

(i) sufficient condition for a given topological space to be quasi-pseudo-metrizable, using of quasi-pseudo-metrization theorem of quasi-uniformity.

(ii) construction of a completion for arbitrary quasi-uniform space.

If $\mathscr{U}$ is a quasi-unifomity on a set X, $\mathscr{U}$ induces a unique topology $\mathscr{T}_u$ on X consisting of all subsets G of X such that for each $x \in G$, $U[x] \subset G$ for some $U \in \mathscr{U}$. We call $\mathscr{T}_u$ the *quasi-uniform topology* induced by $\mathscr{U}$. If $\mathscr{L}$ is a base for $\mathscr{U}$, we call that $\mathscr{U}$ is *generated* by $\mathscr{L}$.

For each function f of X to Y, f_2 is the induced function on $X \times X$ to $Y \times Y$ which is defined by $f_2(x, y) = (f(x), f(y))$. Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be quasi-uniform spaces. A function $f: X \to Y$ is *quasi-uniformly continuous* relative to $\mathscr{U}$ and $\mathscr{V}$, iff $f_2^{-1}[V] \in \mathscr{U}$ for each $V \in \mathscr{V}$. f is *quasi-uniform isomorphism* iff f is one to one onto Y and both f and f^{-1} are quasi-uniformly continuous.

Every quasi-uniformly continuous function is continuous relative to the quasi-uniform topology, hence every quasi-uniform isomorphism is homeomorphism. Consequently, if P is a topological invariant, then P is also a quasi-uniform invariant.

2. Quasi-pseudo-metrization

DEFINITION. A *quasi-pseudo-metric* on a set X is a non-negative real valued function d of $X \times X$, such that

(i) $d(x, x) = 0$,

(ii) $d(x, z) + d(z, y) \geqq d(x, y)$ for all $x, y, z \in X$.

Let d be a quasi-pseudo-metric on X, and $V_{d,r} = \{(x, y) \mid d(x, y) < r\}$.

Unless otherwise specified, we adopt the terminology and notation used by Kelley [4].

Then $\{V_{d,r} | r>0\}$ is a base for a quasi-uniformity $\mathscr{U}$ on X. $\mathscr{U}$ is called the quasi-uniformity generated by quasi-pseudo-metric d. A quasi-uniformity $\mathscr{U}$ on X is *quasi-pseudo-metrizable* iff there exists a quasi-pseudo-metric d on X which generates $\mathscr{U}$.

LEMMA 2.1. *Let $\{U_n \mid n \in \omega\}$, where ω is the set of non-negative integers, be a sequence of subsets of $X\times X$ such that $U_0=X\times X$, each U_n contains Δ_X and $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$ for each $n \in \omega$. Then there exists a quasi-pseudo-metric d on X such that $U_n \subset V_{d,2^{-n}} \subset U_{n-1}$ for each positive integer.* ([4], 185)

We obtain the following:

LEMMA 2.2. *A quasi-uniformity $\mathscr{U}$ on a set X is quasi-pseudo-metrizable iff $\mathscr{U}$ has a countable base.*

It follows immediately that if $\mathscr{U}$ is quasi-pseudo-metrizable, $\mathscr{T}_u$ is quasi-pseudo-metrizable. But the converse does not hold in general. It is explained by the following examples;

EXAMPLE 2.3. Let Ω be the first uncountable ordinal and let $X=\{a \mid a =\text{ordinal}, a<\Omega\}$, $U_a=\{(x, y) \mid x\geqq a, y\geqq a\} \cup \Delta_X$ for each $a \in X$. Then $\mathscr{L}=\{U_a \mid a \in X\}$ is a base for a quasi-uniformity $\mathscr{U}$ on X. It follows that $\mathscr{T}_u$ is quasi-pseudo-metrizable. But $\mathscr{U}$ has no countable base.

EXAMPLE 2.4. Let X be uncountable set and A be a denumerable subset of X, $B_x=A \cup \{x\}$, $\mathscr{L}=\{B_x \mid x \in X\}$ and let $S(x)=B_x\times B_x \cup (X-B_x)\times X$, $\mathscr{S}=\{S(x) \mid x \in X\}$. Then $\mathscr{S}$ is a subbase for a quasi-uniformity $\mathscr{U}$ on X, and $\mathscr{L}$ is a base for a topology $\mathscr{T}$ on X. It follows that $\mathscr{U}$ induces $\mathscr{T}$ by the Lemma 2.5, and that $\mathscr{T}$ is quasi-pseudo metrizable but not pseudo metrizable (Example 2.7). On the other hand, the construction of $\mathscr{S}$ shows that $\mathscr{U}$ has no countable base, hence $\mathscr{U}$ is not quasi-pseudo metrizable.

LEMMA 2.5. *Let $\mathscr{L}$ be a base for a topology $\mathscr{T}$ on X, and let $\mathscr{S}=\{S_B \mid B \in \mathscr{L}\}$, where $S_B=B\times B \cup (X-B)\times X$. Then the quasi-uniformity $\mathscr{U}$ with $\mathscr{S}$ as a subbase induces $\mathscr{T}$.*

PROOF. It is obvious that $S_B \supset \Delta_X$, $S_B \circ S_B = S_B$. To show $\mathscr{T}=\mathscr{T}_u$, let $G \in \mathscr{T}$ and $x \in G$. Then $x \in B \subset G$ for some $B \in \mathscr{L}$, therefore $S_B[x]=B \subset G$.

On the other hand, let $G \in \mathscr{T}_u$ and $x \in G$. Then $\bigcap_{i=1}^{n} S_{B_i}[x] \subset G$ for some $S_{B_i} \in \mathscr{S}$, and hence $G \in \mathscr{T}$.

Now, we give a simple proof of Ribeiro's theorem [6] using of Lemma 2.2 and Lemma 2.5.

THEOREM 2.6. *If $(X, \mathscr{T})$ satisfies the second axiom of countability, then $(X, \mathscr{T})$ is quasi-pseudo metrizable.*

PROOF. Let $\mathscr{L}$ be a countable base for $\mathscr{T}$, and $\mathscr{S} = \{S_B | B \in \mathscr{L}\}$, and let $\mathscr{U}$ is a quasi-uniformity with $\mathscr{S}$ as a subbase. Then $\mathscr{U}$ has a countable base, since $\mathscr{S}$ is countable. Theorefore, $\mathscr{U}$ is quasi-pseudo-metrizable by Lemma 2.2, and hence $\mathscr{T}_u$ is quasi-pseudo metrizable. Lemma 2.5 shows that $\mathscr{T} = \mathscr{T}_u$.

The converse of Theorem 2.6 is not valid.

EXAMPLE 2.7. Let X be a uncountable set and A be a denumerable subset of X, set

$$d(x, y) = \begin{cases} 0 : x=y, \text{ or } x \in A,\ y \in A, \text{ or } x \in X-A,\ y \in A. \\ 1 : x \in A,\ y \in X-A, \text{ or } x \in X-A,\ y \in X-A. \end{cases}$$

Then d is a quasi-pseudo metric but not a pseudo metric, and the topolgy $\mathscr{T}_d$ induced by d has no countable base. Moreover $\mathscr{T}_d$ is not pseudo metrizable.

3. Completeness

In the paper [3], J.L. Sieber and W.J. Pervin left open the question of the existence of a non-trivial completion for an arbitrary quasi-uniform space. It is well known that every uniform space has a completion. This theorem was accomplished using of pseudo-metrics, minimal Cauchy filters, and Cauchy filter bases, respectively. Since quasi-uniformity and quasi-pseudo-metric need not satisfy symmetric law, we can't accomplish the corresponding theorem for quasi-uniform space using of above methods.

R.A. Stoltenberg [5] answered Sieber and Pervin's problem, but in case that a concrete completion for a given quasi-uniform space is needed, Stoltenberg's method is rather complicated. In this section, we answer this problem by a simple construction.

DEFINITION. [3] Let $(X, \mathscr{U})$ be a quasi-uniform space.

(i) A filter $\mathscr{F}$ in $(X, \mathscr{U})$ is *a Caucy filter* iff for each $U \in \mathscr{U}$, there exists $x \in X$ such that $U[x] \in \mathscr{F}$.

(ii) A filter $\mathscr{F}$ in $(X, \mathscr{U})$ *converges* to $x \in X$ iff $\{U[x] \mid U \in \mathscr{U}\} \subset \mathscr{F}$.

(iii) $(X, \mathscr{U})$ is *complete* iff each Cauchy filter in X converges to a point of X.

(iv) A *completion* of $(X, \mathscr{U})$ is a pair $((X^*, \mathscr{U}^*), f)$ such that $(X^*, \mathscr{U}^*)$ is a complete quasi-uniform space and f is a quasi-uniform isomorphism of $(X, \mathscr{U})$ onto a dense subspace of $(X^*, \mathscr{U}^*)$.

THEOREM 3.1. Let $(X, \mathscr{U})$ be a quasi-uniform space, and $\mathfrak{a}$ be an arbitrary cardinal number so that $\mathfrak{a} \geqq |X|$[1]. Then there exists a quasi-uniform completion $((X^*, \mathscr{U}^*), f)$ of $(X, \mathscr{U})$ such that $|X^*| = \mathfrak{a}$.

PROOF. Let A be a disjoint set from X such that $|A| = \mathfrak{a}$, and let $X^* = X \cup A$. Setting $\mathscr{L} = \{U \cup A \times X^* \mid U \in \mathscr{U}\}$, it follows that $\mathscr{L}$ is a base for a quasi-uniformity on X^*. For,

(a) $U \cup A \times X^*$ contains the diagonal of X^*,

(b) $(U \cup A \times X^*) \cap (V \cup A \times X^*) = (U \cap V) \cup A \times X^*$,

(c) for each $U \cup A \times X^* \in \mathscr{L}$, choose $V \in \mathscr{U}$ such that
$V \circ V \subset U$, then $(V \cup A \times X^*) \circ (V \cup A \times X^*) \subset U \cup A \times X^*$.

Let $\mathscr{U}^*$ be the quasi-uniformity on X^* generated by $\mathscr{L}$. The construction of $\mathscr{U}^*$ shows that $(X^*, \mathscr{U}^*)$ is a complete quasi-uniform space. In fact, every Cauchy filter in $(X^*, \mathscr{U}^*)$ converges to an arbitrary point of A. For each $U^* \in \mathscr{U}^*$, $U^* \cap X \times X \in \mathscr{U}$ and hence subspace $X^* - A$ of X^* is $(X, \mathscr{U})$, that is, $(X, \mathscr{U}) = (X, \mathscr{U}^* \cap X \times X)$. Therefore, $(X, \mathscr{U})$ is quasi-uniformly isomorphic to $(X^*, \mathscr{U}^*)$ by identity mapping f. It is clear that the closure of $X^* - A$ is X^*. Thus $((X^*, \mathscr{U}^*), f)$ is a completion of $(X, \mathscr{U})$, and $|X^*| = \mathfrak{a}$.

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(1) $|X|$ denotes the cardinal number of X.

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