ON BILATERAL CALCULUS IN TWO VARIABLES I

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1. In this paper, I have established a Sheorem in bilateral calculus to obtain certain bilateral operational relations and also evaluated the infinite integrals which show the possible application of the theorem established. The domain of convergence has been defined in Die Zwei Dimensionale Laplace transform by Doetsch and Voelker [1] and by Ditkin and Prudnikov [2] in operational calculus in two variables and its applications. It has also been investigated by me in my thesis for doctor's degree. So I have made no attempt here to discuss the region of convergence but I have indicated the conditions under which this theorem is valid in my enunciations of the theorem. The results are believed to be new.

2. THEOREM 1. *Let*

(i) $p_1p_2F(p_1, p_2) \risingdotseq G(s_1, s_2)$, $1/p_r \risingdotseq s_r$ $(r=1, 2)$, *where* $L_\pi^2\{G\}$ *is absolutely convergent in a domain* s_{p_i} *defined by* $\alpha_i \leq R(p_i) \leq \beta_i$ *where* $i=1,2$.

(ii) $p_r[\phi_r(p_r)]^k \exp\left[-s_r\phi_r{}^l(p_r)+\frac{1}{\phi_r{}^m(p_r)}\right] \risingdotseq \theta_r(s_r, x_r)$, $1/p_r \risingdotseq x_r$, $r=1,2$ *in which* s_r *occurs as a real parameter and* $L\pi\{\theta_r\}$ *is absolutely convergent in* $(-\infty, \infty)$ *with respect to* x_r *in the associated convergence domains* $D_{p_r}(r=1, 2)$ *and* $\phi_r{}^l(p_r)+\frac{1}{\phi_r{}^m(p_r)} \epsilon S_{p_r}$.

(iii) $\theta_r(s_r, x_r)$ *is bounded and abbsolutely integrable in* $(-\infty, \infty)$ *or in* $(-S_r \leq s_r \leq S_r)$.

(iv) $G(s_1, s_2)$ *is absolusely integrable in* s_r, *in* $(-\infty, \infty)$. *Then*

$$p_1p_2[\phi_1(p_1)\ \ \phi_2(p_2)]^k F\left\{\phi'_1(p_1)+\frac{1}{\phi_1{}^m(p_1)},\ \phi_2{}^l(p_2)+\frac{1}{\phi_2{}^m(p_2)}\right\}$$

$$\risingdotseq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta_1(s_1, x_1)\ \theta_2(s_2, x_2)\ G(s_1, s_2)\ ds_1\, ds_2, \tag{2.1}$$

provided that the definition integral is absolutely convergent in a pair of associated convergence domains, say Ω_{p_1} *and* Ω_{p_2} *where* Ω_{p_1} *is the common region of* D_{p_1} *and* S_{p_1} *and* Ω_{p_2} *is the common region of* D_{p_2} *and* S_{p_2}. *The conditions* (iii) *and (iv) can be waived and replaced by the single condition that the integral*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-p_1x_1-p_2x_2}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1(s_1, x_1)\theta_2(s_2, x_2)G(s_1, s_2)\, ds_1 ds_2\right\}dx_1\, dx_2,$$

is *absolutely convegent.*

PROOF We have

$$F(p_1, p_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-p_1s_1-p_2s_2}\, G(s_1, s_2)\, ds_1 ds_2. \tag{2.2}$$

On replacing p_r by $\phi_r^{\,l}(p_r)+\dfrac{1}{\phi_r^{\,m}(p_r)}$ $(r=1,2)$ in (2. 2), we get

$$F\left\{\phi_1^{\,l}(p_1)+\frac{1}{\phi_1^{\,m}(p_1)},\ \phi_2^{\,l}(p_2)+\frac{1}{\phi_2^{\,m}(p_2)}\right\}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s_1\left[\phi_1^{l}(p_1)+\frac{1}{\phi_1^{m}(p_1)}\right]}e^{-s_2(\phi_2^{l}(p_2)+\phi_2^{-m}(p_2))}G(s_1, s_2)ds_1 ds_2 \tag{2.3}$$

On multiplying both sides of (2.3) by $p_1p_2\ [\phi_1(p_1)\ \phi_2(p_2)]^k$ which is permissible, we get

$$p_1p_2\ [\phi_1(p_1)\ \phi_2(p_2)]^k\ F\left\{\phi_1^{\,l}(p_1)+\frac{1}{\phi_1^{\,m}(p_1)},\ \phi_2^{\,l}(p_2)+\frac{1}{\phi_2^{\,m}(p_2)}\right\}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_1[\phi_1(p_1)]^k\ e^{-s_1\left(\phi_1^{l}(p_1)+\frac{1}{\phi_1^{m}(p_1)}\right)}\ p_2[\phi_2(p_2)]^k\ e^{-s_2(\phi_2^{l}(p_2)+\phi_2^{-m}(p_2))}$$

$$\times G(s_1, s_2)\ ds_1 ds_2$$

$$=p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(s_1, s_2)\ ds_1ds_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-p_1x_1-p_2x_2}\,\theta_1(s_1, x_1)\theta_2(s_2, x_2)dx_1dx_2, \tag{2.4}$$

valid in the pair of associated convergence domains Ω_{p_1} and Ω_{p_1} as defined above. Since the x-integrals and the s-integrals are absolutely convergent due to (ii), (iii) and (iv), a change of order of integration in (2.4) is permissible. Therefore the right hand side of (2.4) is

$$p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-p_1x_1-p_2x_2}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_1(s_1, x_1)\ \theta_2(s_2, x_2)\ G(s_1, s_2)\ ds_1ds_2\right\}dx_1dx_2.$$

and we get the theorem desired.

3. Corollaries based on the above theorem

(a) Let $k=-1/2$, $l=-1/2$, $m=-1/2$ and $\phi_r(p_r)=p_r=\phi_r(p_r)$

Hence from the theorem, we get

$$\sqrt{p_1p_2}\ F\left[2\sqrt{p_1},\ 2\sqrt{p_2}\right] \doteqdot \frac{U(x_1)\ U(x_2)}{\pi\sqrt{x_1x_2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s_1/x_1-s_2/x_2} G(s_1,\ s_2)\ ds_1ds_2.$$

(b) Let $k=l=1,\ \ m=-1,$ and $\phi_r(p_r)=\sqrt{p_r^2+a^2}-p_r$

$\phi_r(p)_r=(p_r^2+a^2)^{-\frac{1}{2}}\left(p_r+\sqrt{p_r^2+a^2}\right)^{-v}.$

Hence from the theorem, we get

$$\frac{p_1p_2\ (p_1^{\,2}+a^2)^{-1/2}\ (p_2^2+a^2)^{-1/2}}{(p_1+\sqrt{p_1^{\,2}+a^2})^v\ p_2+\sqrt{(p_2^2+a^2)}^v}\ F\left[2(\sqrt{p_1^{\,2}+a^2}-p_1,\ 2(\sqrt{p_2^{\,2}+a^2}-p_2)\right]$$

$$\doteqdot a^{-2v}\ U(x_1)\ U(x_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1x_2)^{v/2}\ [(x_1+4s_1)\ (x_2+4s_2)]^{-v/2}$$

$$\times J_v\left[a\sqrt{x_2^{\,2}+4s_2x_2}\right]\ J_v\left[a\sqrt{x_1^{\,2}+4s_1x_1}\right]\ G(s_1,\ s_2)\ ds_1ds_2.$$

(c) Let $k=l=1,\ \ m=\infty,\ \ \phi_r(p_r)=p_r^{\ -\mu_r},\ \ \phi_r(p_r)=p_r^{\ -\lambda_r+1}$

$$p_r^{\ -\lambda_r}\ e^{-s_r/p_r^{\mu_r}} \doteqdot x_r^{\ \lambda_r}\ J_{\lambda_r}^{\mu_r}(s_rx_r^{\ \mu_r})\ U(x_r),\ \ r=1,2.$$

Hence from the theorem, we get

$$p_1^{-\lambda_1}\ p_2^{-\lambda_2}\ F[p_1^{\ -\mu_1},\ p_2^{\ -\mu_2}] \doteqdot U(x_1)\ U(x_2)\ x_1^{\lambda_1}\ x_2^{\lambda_2}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{\lambda_1}^{\mu_1}(s_1x_1^{\ \mu_1})\ J_{\lambda_2}^{\mu_2}(s_2x_2^{\ \mu_2})\ G(s_1,\ s_2)\ ds_1\ ds_2. \qquad (c_1)$$

In particular, if we take $\mu_1=\mu_2=1,$ we get

$$p_1^{-\lambda_1}\ p_2^{-\lambda_2}\ F(1/p_1,\ 1/p_2) \doteqdot U(x_1)\ U(x_2)\ x_1^{\lambda_1/2}\ x_2^{\lambda_2/2}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_1^{\lambda_1/2}\ s_2^{\lambda_2/2}\ J_{\lambda_1}\left(2\sqrt{s_1x_1}\right)\ J_{\lambda_2}\left(2\sqrt{s_2x_2}\right) G(s_1,\ s_2)\ ds_1ds_2. \qquad (c_2)$$

4. Examples

(d) Let $p_1p_2\ F(p_1,\ p_2)=\dfrac{p_1p_2}{p_1^2-p_2^2} \doteqdot -1/2\ U(s_2-|s_1|)=G(s_1,\ s_2)$

From Cor. (c), we get (when $\lambda_1=\lambda_2=1$)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (s_1s_2)^{-1/2}\ J_1\left(2\sqrt{s_1x_1}\right)\ J_1\left(2\sqrt{s_2x_2}\right)\ U(s_2-|s_1|)\ ds_1ds_2=\frac{U(x_1-|x_2|)}{x_1x_2}$$

(e) Let $\ pF(p)=p^{1-v}\ I_v(p) \doteqdot \dfrac{(1-s^2)^{v-1/2}}{\pi_2^{\ v}\ \Gamma(v+1/2)}U(1-s^2)\equiv G(s)$

$$p^{-\lambda}\ F(1/p)=p^{1-\lambda}I_v(1/p)$$

$$\doteqdot 2^{-v}x^{\lambda}[(\Gamma(v+1)\Gamma(\lambda+1)]^{-1}\ {}_0F_3\left(v+1,\ \frac{\lambda+1}{2},\ \frac{\lambda}{2}+1;\frac{x^2}{16}\right)U(x).$$

Hence from Cor. (c), we get

$$\int_{-\infty}^{\infty}\frac{J_\lambda(2\sqrt{sx})\ U(1-s^2)}{s^{\lambda/2}\ (1-s^2)^{1/2-v}}\,ds=\frac{x^{\lambda/2}\ \pi\Gamma(v+1/2)}{\Gamma(v+1)\ \Gamma(\lambda+1)}\,{}_0F_3\left(v+1,\ \frac{\lambda+1}{2},\frac{\lambda}{2}+1;\frac{x^2}{16}\right),$$

$R(v)> 1/2$.

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