TOTALLY UMBILICAL HYPERSURFACES OF THE ALMOST EINSTEIN SPACE

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Introduction

Recently in a locally product Riemannian manifold, M. Okumura [1] (*) tried to prove the totally umbilical hypersurface with constant mean curvature to be isometric with a sphere and obtained some results about the totally umbilical hypersurfaces with constant mean curvature.

In the totally umbilical hypersurface with the non-constant mean curvature, however, there does not exist above mentioned ones. In the present paper, the author tries to discuss the properties of the totally unbilical hypersurfaces with non-constant mean curvature.

We shall devote § 1, to preliminaries. In § 2, we obtain the mean curvature and scalar curvature of the totally umbilical hypersurfaces in the special kind of locally produt Riemannian manifolds. In § 3, we find some properties of the totally umbilical hypersurfaces in the almost Einstein space.

§ 1. Preliminaries

Let there be given an $(n+1)$-dimensional locally product orientable Riemannian manifold $\overline{M}^{n+1}$ with locally coordinates $\{X^\lambda\}$, Then, by definition, there exist a sys tem of coordinate neighbourhoods $\{U\alpha\}$ such that in each $U\alpha$ the first fundamental form of $\overline{M}^{n+1}$ is given by

$$(1.1) \qquad ds^2 = \sum_{a,b=1}^{p} G_{ab}(X^c)dX^a dX^b + \sum_{t,s}^{q} G_{st}(X^r)dX^s dX^t,$$

$$p \geqq 2, \quad q \geqq 2 \qquad p+q=n+1,$$

and in $U_{\alpha \cap} U_\beta$ the coordinate transformation $(X^a,\ X^t) \rightarrow (X^{a'},\ X^{t'})$ is givn by

$$X^{a'} = X^{a'}(X^a), \quad X^{t'} = X^{t'}(X^t).$$

If we define F^λ_μ by

$$(1.2) \qquad (F^\lambda_\mu) = \begin{pmatrix} \delta^a_b & 0 \\ 0 & -\delta^s_t \end{pmatrix},$$

(*) Numbers in brackets refer to references at the end of the paper.

$$a,\ b=1,\ 2,\cdots\cdots,\ p, \qquad s,\ t=1,\ 2,\cdots\cdots,\ q,$$

in each $U\alpha$, then F_μ^λ is a tensor field over $\overline{M}^{n+1}$ and satisfies

(1.3) $F_\mu^\lambda F_\nu^\mu=\delta_\nu^\lambda,$

(1.4) $F_{\lambda\mu}=G_{\nu\mu}F_\lambda^\nu=F_{\mu\lambda},$

(1.5) $\bar{\nabla}_\mu F_\nu^\lambda=0,$

where $\bar{\nabla}_\mu$ denote the operator of the Riemannian covariant derivative with respect to $G_{\nu\mu}$, (1.3) shows that F_μ^λ assigns an almost product structure of $\overline{M}^{n+1}$.

Let M^n be orientable hypersurface of $\overline{M}^{n+1}$ reprsented parametically by

$$X^\lambda=X^\lambda(x^h),$$

where $\{x^h\}$ are locally coordinates of M^n. We put $B_i^\lambda=\partial X^\lambda/\partial x_i$, then B_i^λ $(i=1, 2, 3,\cdots\cdots, n)$ are linearly independent tangent vectors at each point of M^n.

The induced Riemannian metric g_{ji} of M^n is given by

(1.6) $g_{ji}=G_{\lambda\mu}B_j^\lambda B_i^\mu.$

If C_λ are the contravariant components of the unit vector normal to M^n, then we get

(1.7) $G_{\lambda\mu}B_h^\lambda C^\mu=0,\quad G_{\lambda\mu}C^\lambda C^\mu=1.$

The transformations $F_\mu^\lambda B_i^\mu$ and $F_\mu^\lambda C^\mu$ can be expressed as linear combinations of B_i^λ and C_i^λ. We put

(1.8) $F_\mu^\lambda B_i^\mu=f_i^h B_h^\lambda+f_i C^\lambda,$

(1.9) $F_\mu^\lambda C^\mu=f^h B_h^\lambda+fC^\lambda,$

from which we have obviously

(1.10) $f_i^h=F_\mu^\lambda B_\lambda^h B_i^\mu,$

(1.11) $f_i=F_\mu^\lambda B_i^\mu C_\lambda,$

(1, 12) $f=F_\mu^\lambda C_\lambda C^\mu,$

where we denote by $(B_\lambda^i,\ C_\lambda)$ the dual basis of $(B_i^\lambda,\ C^\lambda)$.

Moreover we have the following

(1.13) $f_{ji}=g_{ih}f_j^h=f_{ij},$

(1.14) $f_i^h f_h^j=\delta_i^j-f^j f_i,$

(1.15) $f_i^h f_h = -f f_i, \qquad f^i f_i^h = -f f^h,$

(1.16) $f_i f^i = 1 - f^2,$

(1.17) $f_i^i = (p-q) - f.$

If we denote ∇_j the operator of covariant differentiation along the M^n, we have the equation of Gauss(*)

(1.18) $$\nabla_j B_i^\lambda \overset{\text{def}}{=\!=} \partial_j B_i^\lambda + B_j^\mu B_i^\nu \overline{\left\{ {\lambda \atop \mu\nu} \right\}} - B_h^\lambda \left\{ {h \atop ji} \right\} = H_{ji} C^\lambda,$$

where $\left\{ {h \atop ji} \right\}$ and $\overline{\left\{ {k \atop \mu\lambda} \right\}}$ are the Christoffel symbols formed from g_{ji} and $G_{\mu\lambda}$ respectively, and H_{ji} are components of the second fundamental tensor of M^n.

We have also the equation of Weingarten

(1.19) $$\nabla_j C^k = \partial_j C^k + B_j^h C^\nu \overline{\left\{ {\lambda \atop \mu\nu} \right\}} = -H_j^i B_i^\lambda,$$

where $H_j^i = g^{ih} H_{jh}$.

Differentiating (1.8) and (1.9) covariantely along M^n and transvecting with B_{ek} and C_k respectively, we get the following relations

(1.20) $\nabla_j f_i^h = f^h H_{ji} + f_i H_j^h,$

(1,21) $\nabla_j f_h = f H_{jh} - f_{ih} H_j^i,$

(1.22) $\nabla_j f = -2 f^h H_{jh}.$

§ 2. Scalar curvature of M^n.

Let M^n be an n-dimensional totally umbilical hypersurface of locally product Riemannian manifold, then it satisfies

(2.1) $H_{ji} = H g_{ji}$

at each point of the M^n, where $H = \frac{1}{n} H_{ji} g^{ji}$ is mean curvature and it is scalar.

From the Codazzi's equation of the M^n

(2.2) $$\nabla_j H_{ih} - \nabla_i H_{jh} = B_j^\nu B_i^\mu B_h^\lambda C^\beta \overline{R}_{\nu\mu\lambda\beta},$$

where $\overline{R}_{\nu\mu\lambda\beta}$ are components of curvature tensor in $\overline{M}^{n+1}$, we have

$$g_{ih} \nabla_j H - g_{jh} \nabla_i H = B_j^\nu B_i^\mu B_h^\lambda C^\beta \overline{R}_{\nu\mu\lambda\beta}.$$

(*) He call $\nabla_j B_i^k$ the van der Waerden-Bortolotti covariant derivavite of B_i^k along the hypersurface. [3]

Transvecting with g^{ih}, we get

$$(2.3)\qquad (n-1)\nabla_j H=(G^{\mu\lambda}-C^{\mu}C^{\lambda})B_j^{\nu}C^{\beta}\overline{R}_{\nu\mu\lambda\beta}$$
$$=\overline{R}_{\nu\beta}B_j^{\nu}C^{\beta}.$$

On the other hand, a locally product Riemannian space is called an almost Einstein space [2], if its Ricci tensor has the following form

$$(2.4)\qquad \overline{R}_{\nu\beta}=\alpha G_{\nu\beta}+\beta F_{\nu\beta},$$

where α, β being necessarily constant.
Since, in an almost Einstein space, it holds

$$(2.5)\qquad \nabla_j H=af_i,$$

where $a=\beta/(n-1)$.
We can prove the

LEMMA 1. *In totally umbilical hypersurface of almost Einstein space, $H\fallingdotseq$ constant.*

PROOF. If H=constant, since $\overline{M}^{n+1}$ be almost Einstein space, from (2.5) we have $f_j=0$ and so $f=\pm 1$ because of (1.16).

On the other hand, making use of (1.21), we have $f_{ih}=fg_{ih}$.
These imply that

$$(2.6)\qquad f_{ih}=\pm g_{ih}.$$

Substituting (2.6) into (1.8) and (1.9) and regarding that $f_j=0$,

we have
$$F_{\mu}^{\lambda}B_i^{\mu}=\pm B_i^{\lambda},$$
$$F_{\mu}^{\lambda}C^{\mu}=\pm C^{\lambda},$$

from which we obtain

$$F_{\mu}^{\lambda}=\pm\delta_{\mu}^{\lambda}.$$

This contradicts the fact F_{μ}^{λ} is non-trivial almost product stucture $\overline{M}^{n+1}$. So H can not be constant over M^n.

Next, we assume that $\overline{M}^{n+1}$ is Einstein locally product space, then $\nabla_j H=0$.
Thus we have the

LEMMA 2. *In totally umbilical hypersurface of Einstein locally product space, H=constant.*

In totally umbilical hypersurface M^n of locally product Riemannian manifold, we get the following

$$(2.7)\qquad \nabla_j f_i^h=H(f^h g_{ji}+f_i\delta_j^h),$$

(2.8) $\nabla_j f_h = (f g_{jh} + f_{jh})$.

(2.9) $\nabla_j f = -2H f_j$.

Let R_{ljih} be covariant components of curvature tensor of M^n, then from Gauss' equation for M^n in almost Einstein space $\overline{M}^{n+1}$

(2.10) $R_{ljih} = B_l^\nu B_j^\mu B_i^\lambda B_h^\beta \overline{R}_{\nu\mu\lambda\beta} + H^2(g_{ji} g_{lh} - g_{li} g_{jh})$.

Transvecting with g^{lh}

(2.11) $R_{ji} = \alpha g_{ji} + \beta f_{ji} - B_j^\mu B_i^\lambda C^\nu C^\beta \overline{R}_{\nu\mu\lambda\beta} + H^2(n-1) g_{ji}$,

and moreover transvecting with g^{ji}, we have

(2.12) $R = (n-1)(\alpha + nH^2) + \beta(p - q - 2f)$.

Thus, from LEMMA 2, we have

THEOREM 2.1. *In totally umbilical hypersurface of Einstein locally product space,* $R = (n-1)\ (\alpha + nH^\lambda)$ *is constant.*

§ 3. Totally umbilical hypersurfaces of almost Einstein and Einstein locally product space.

In totally umbilical hypersurface M^n of Einstain locally product space $\overline{M}^{n+1}$, we have H=constant and

(3.1) $\nabla_j \nabla_i f = 2H^2(f_{ji} - f g_{ji})$.

Again, differentiating covariantly, we get

(3.2) $\nabla_j \nabla_i \nabla_h f = -H^2(2g_{ih}\nabla_j f + g_{jh}\nabla_i f + g_{ji}\nabla_h f)$,

and(*)

(3.3) $\underset{\nabla f}{\pounds} g_{ji} = \nabla_j \nabla_i f + \nabla_i \nabla_j f$

$= 4H^2(f_{ji} - f g_{ji})$.

Hence, in totally umbilical hypersurface M^n of Einstein locally product space $\overline{M}^{n+1}$, we get $\underset{\nabla f}{\pounds} g_{ji} \neq 0$, i.e. $\nabla_j f$ does not become the Killing in M^n.

In fact, if $\underset{\nabla f}{\pounds} g_{ji} = 0$, from (3.3) we have

$f_{ji} - f g_{ji} = 0$.

Transvecting with g^{ji}, then we have

(*) $\underset{\nabla f}{\pounds}$ denotes the operator of Lie derivation with respect to $\nabla_j f$

$$f=\frac{p-q}{n+1}, \text{ i.e., } f \text{ is constant.}$$

This contradicts the fact that f can not be constant. So $\underset{vf}{\pounds} g_{ji} \neq 0$.

From (3.2) and Ricci's identity with respect to $\nabla_h f$, we have

$$\text{(3.4)} \qquad \underset{vf}{\pounds}\left\{{h \atop ji}\right\}=\nabla_j\nabla_i\nabla^h f+R_{kji}{}^h\nabla^k f$$
$$=-2H^2(\delta_i^h\nabla_j f+\delta_j^h\nabla_i f).$$

Substituting (3.4) into the following identity

$$\text{(3.5)} \qquad \underset{vf}{\pounds} R_{lji}{}^h=\nabla^l\left(\underset{vf}{\pounds}\left\{{h \atop ji}\right\}\right)-\nabla^j\left(\underset{vf}{\pounds}\left\{{h \atop li}\right\}\right),$$

we have

$$\text{(3.6)} \qquad \underset{vf}{\pounds} R_{lji}{}^h=-2H^2(\delta_l^h\nabla_l\nabla_i f-\delta_l^h\nabla_j\nabla_i f).$$

And contracting h and l

$$\text{(3.7)} \qquad \underset{vf}{\pounds} R_{ji}=C\underset{vf}{\pounds} g_{ji},$$

where $C=(n-1)H^2$ is constant. Thus we have

THEOREM 3.1. *In totally umbilical hypersurfaces M^n of Einstein locally product space $\overline{M}^{n+1}$, $\underset{vf}{\pounds} R_{ji}=C\underset{vf}{\pounds} g_{ji}$.*

From (3.7), We have

$$\text{(3.8)} \qquad (\nabla^a f)\nabla_a R_{ji}+R_{ai}\nabla_j\nabla^a f+R_{ja}\nabla_i\nabla^a f$$
$$-cg_{ai}\nabla_j\nabla^a f-cg_{ja}\nabla_i\nabla^a f=0.$$

Transvecting with g^{ji}, it holds

$$\text{(3.9)} \qquad R_{ji}\nabla^j\nabla^i f=c\Delta f,$$

because of $g^{ji}\nabla_a R_{ji}=\nabla_a R=0$,

where $\qquad \Delta f=g_{ji}\nabla^j\nabla^i f.$

If we assume that M^n is compact, from Green's theorem, we have

$$\text{(3.10)} \qquad \int_{M^n} R_{ji}\nabla^j\nabla^i f d\sigma=0.$$

where $d\sigma$ is volume element.

And in such M^n, if $g^{ji}\nabla_j\nabla_i f \geqq 0$ for scalar f, then f is constant [4].

Therfore we know that $g^{ji}\nabla_j\nabla_i f<0$.

From (3.1), we have

$$f>\frac{p-q}{n+1}. \tag{3.11}$$

Thus we conclude

THEOREM 3.2. *In compact orientable totally umbilical hypersurfaces of Einstein locally product space,*

$$\int_{M^n} R_{ji}\nabla^j\nabla^i f d\sigma=0 \quad \text{and} \quad f>\frac{p-q}{n+1}.$$

Now consider that $\bar{M}^{n+1}$ is an almost Einstein space, from LEMMA 1, $H \neq$ constant and

$$\nabla_j\nabla_i f=-2af_jf_i+2H^2(f_{ji}-fg_{ji}). \tag{3.12}$$

where $a=\frac{\beta}{n-1}$,

moreover

$$\underset{\nabla f}{\pounds} g_{ji}=-4af_jf_i+4H^2(f_{ji}-fg_{ji}). \tag{3.13}$$

Again differentiating (3.12) covariantly, we get by (2.7), (2.8)

$$\begin{aligned}\nabla_j\nabla_i\nabla_h f=&a[(fg_{ji}-f_{ji})\nabla_h f+2(fg_{ih}-f_{ih})\nabla_j f+(fg_{hj}-f_{hj})\nabla_i f]\\&-H^2(g_{ji}\nabla_h f+2g_{ih}\nabla_j f+g_{hj}\nabla_i f)\end{aligned} \tag{3.14}$$

from which and the Ricci's identity with respect to $\nabla_h f$ we have

$$\begin{aligned}\underset{\nabla f}{\pounds}\left\{{h \atop ji}\right\}&=\nabla_j\nabla_i\nabla^h f+R_{lji}{}^h\nabla^l f\\&=-2a(f_j^h\nabla_i f+f_i^h\nabla_j f)+\delta_j^h\nabla_i\rho+\delta_i^h\nabla_j\rho\end{aligned} \tag{3.15}$$

where $\rho_j=\nabla_j[-2(af+H^2)f]$ are gradiant.

Substituting (3.15) into (3.5), we have

$$\begin{aligned}\underset{\nabla f}{\pounds} R_{lji}{}^h=&-2a[(\nabla_j f)\nabla_l f_i^h+f_j^h\nabla_l\nabla_i f+(\nabla_i f)\nabla_l f_j^h]\\&+\delta_j^h\nabla_l\rho_i+2a[\nabla_l f)\nabla_j f_i^h+f_l^h\nabla_j\nabla_i f+(\nabla_i f)\nabla_j f_l^h]\\&-\delta_l^h\nabla_j\rho_i.\end{aligned} \tag{3.16}$$

contracting h and l and by (2.7), (2.8), and (2.9),

$$\begin{aligned}\underset{\nabla f}{\pounds} R_{ji}=&(n-1)(af+H^2)\underset{\nabla f}{\pounds} g_{ji}+8aH^2[nf_jf_i-(1-f^2)g_{ji}]\\&+2(af_jf_i-H^2f_{ji}+H^2fg_{ji})(p-q).\end{aligned} \tag{3.17}$$

From (3.13), we get

(3.18) $\underset{\nabla f}{\pounds} R_{ji} = [(n-1)(af+H^2) - \frac{1}{2}(p-q)] \underset{\nabla f}{\pounds} g_{ji} + 8aH^2[nf_j f_i - (1-f^2)g_{ji}]$,

and by (1.16)

(3.19) $g^{ji} \underset{\nabla f}{\pounds} R_{ji} = A g^{ji} \underset{\nabla f}{\pounds} g_{ji}$,

where $A = (n-1)(af+H^2) - \frac{1}{2}(p-q)$ is constant.

Hence if $\underset{\nabla f}{\pounds} R_{ji} = 0$, then $\nabla^j \nabla_i f = 0$, therefore $\nabla_j f$ is harmonic vector. Thus we have

THEOREM 3.3. *In totally umbilical hypersurface of almost Einstein space, if* $\underset{\nabla f}{\pounds} R_{ji} = 0$, *then* $\nabla_j f$ *is a harmonic vector, and*

$$H^2 = \frac{a(f^2-1)}{(n+1)f}$$

From (3.19)

$$R_{ji} \nabla^j \nabla^i f = A \Delta f$$

Thus we have

THEOREM 3.4. *In compact onentable totally umbilical hypersurface* M^n *of almost Einstein space*

$$\int_{M^n} R_{ji} \nabla^j \nabla^i f \delta\sigma = 0 \text{ and } g^{ji} \nabla_j \nabla_i f < 0$$

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