$\Pi_1(S^N)=0$

By Larry Smith

The usual proof that $\Pi_1(S^N)=0$ for $N>1$ depends upon the simplicial approximation theorem [1]. As the problem is strictly a homotopy problem this is somewhat unpleasent. In this note we give a simple proof using only the existance of the universal covering space; the usual device for showing $\Pi_1(S^1)=Z$.

We shall need the following basic lifting property of covering spaces:

PROPOSITION *If* $\tilde{X}\xrightarrow{\pi}X$ *is a covering map and* $f:(I^N,\partial I^N)\longrightarrow(X,x_0)$ *is a map of pairs, then there exists a map of pairs* $\tilde{f}:(I^N,\ \partial I^N)\xrightarrow{\pi}(\tilde{X},\ \pi^{-1}(x_0))$ *making the diagram*

$$\begin{array}{ccc} & & (\tilde{X},\ \pi^{-1}(x_0)) \\ & \tilde{f}\nearrow & \downarrow\pi \\ (I^N,\ \partial I^N) & \xrightarrow{f} & (X,\ x_0) \end{array}$$

commute.

This may be readily proved by induction on N using the covering homotopy property and the trick of regarding a map $f:I^N\longrightarrow X$ as a homotopy from $f|I^{N-1}\times\{0\}\longrightarrow X$ to $f|\ I^{N-1}\times\{1\}\longrightarrow X$.

Since $\pi^{-1}(x_0)$ is discrete and ∂I^N is connected for $N>1$ it follows that for $N>1$ $f(\partial I^N)\subset\tilde{x}_0\ \varepsilon\ \pi^{-1}(x_0)$. Recalling that $S^N=I^N/\partial I^N$ we obtain:

COROLLARY *If* $\tilde{X}\xrightarrow{\pi}X$ *is a covering map and* $f:(S^N,\ p)\longrightarrow(X,\ x_0)$, $N>1$, *then there exists* $\tilde{f}:(S^N,p)\longrightarrow(\tilde{X},\ \tilde{x}_0)$ *making the diagram*

$$\begin{array}{ccc} & & (\tilde{X},\ \tilde{x}_0) \\ & \tilde{f}\nearrow & \downarrow\pi \\ (S^N,\ p) & \xrightarrow{f} & (X,\ x_0) \end{array}$$

commute.

PROPOSITION $\Pi_1(S^N)=0$ *for* $N>1$.

PROOF, Let $U^N \longrightarrow S^N$ be the universal covering space of S^N. Applying the corollary to the map $1 : S^N \longrightarrow S^N$ gives a diagram

$$\begin{array}{ccc} & & U^N \\ & \nearrow & \downarrow \\ S^N & \xrightarrow{1} & S^N \end{array}$$

and passing to fundamental groups gives

$$\begin{array}{ccc} & & \Pi_1(U^N)=0 \\ & \nearrow & \downarrow \\ \Pi_1(S^N) & \xrightarrow{1} & \Pi_1(S^N) \end{array}$$

and hence $\Pi_1(S^N)=0$.

Princeton University

REFERENCES

[1] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press 1952.

[2] W. S. Massey, *Introduction to Algebraic Topology*, Harcourt Brace and World, 1967.