A CHARACTERIZATION OF COMPACT CORRECT UNIFORM SPACES

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THEOREM 1. *A compact symmetric generalized uniform space (c.f.* [2]) $(X,\mathscr{U})$ *is totally bounded.*

PROOF Immediate consequence of theorem 4 in [2].

THEOREM 2. *Let* $(X, \mathscr{U})$ *be a symmetric generalized uniform space that has the property that if* $U\in\mathscr{U}$ *then there exists* V *in* $\mathscr{U}$ *such that* $V\circ V\subset U$. *Then* $(X, \mathscr{U})$ *is compact if it is complete and totally bounded.*

LEMMA 1. *If a Cauchy filter in* $(X, \mathscr{U})$ *has a cluster point, it converges to it.*

LEMMA 2. *If* $(X, \mathscr{U})$ *is totally bounded, then every ultrafilter is a Cauchy filter.*

The proofs of lemma 1 and lemma 2 are very similar to those for an ordinary uniform space. (c.f. [3] page 186).

PROOF OF THEOREM 2 Suppose $(X, \mathscr{U})$ is compact. Then (by theorem 1) $(X, \mathscr{U})$ is totally bounded. Also, every filter and hence every Cauchy filter has a cluster point; therefore (by lemma 1) $(X, \mathscr{U})$ is complete. Conversely, let $\mathscr{F}$ be an ultrafilter on X. Then (by lemma 2) $\mathscr{F}$ is a Cauchy filter and hence converges since $(X, \mathscr{U})$ is complete. Hence $(X ,\mathscr{U})$ is compact.

COROLLARY *Let* $(X, \mathscr{U})$ *be a uniform space.* $(X, \mathscr{U})$ *is compact iff it is complete and totally bounded.*

PROOF This is an immediate consequence of theorem 2 and the fact that every uniform space has a symmetric base.

DEFINITION (Efremovic et. al. [1]) A Cauchy filter $\mathscr{F}$ is an *infrafilter* iff there is no Cauchy filter $\mathscr{F}_1$ such that $\mathscr{F}_1$ is properly contained in $\mathscr{F}$.

THEOREM 3. *Let $(X, \mathscr{U})$ be a correct uniform space (c.f.* [1]*). Then $(X, \mathscr{U})$ is compact iff it is totally bounded and every infrafilter in $(X, \mathscr{U})$ is a neighborhood filter of some point.*

LEMMA 3. *$(X, \mathscr{U})$ is complete implies every infrafilter is a neighborhood filter of some point.*

LEMMA 4. *Every Cauchy filter in $(X, \mathscr{U})$ contains an infrafilter. The above two lemmas are proved in* [1].

PROOF OF THEOREM 3. We have (by lemma 3 and lemma 4) that $(X, \mathscr{U})$ is complete iff every infrafilter in it is a neighborhood filter of some point. The theorem is now a direct consequence of theorem 2.

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REFERENCES

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[3] W. J. Thron, *Topological Structures,* Holt, Rinehart, and Winston, New York, 1966.