NOTE ON THE CHARACTERIZATIONS OF MINIMAL T_0 AND T_D SPACES

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Introduction

Given a set X and the lattice of all topologies on the set. We will investigate properties of the minimal T_0 topology and minimal T_D topology on this set. In the paper [4], M.P. Berri discussed minimal properties of Hausdorff spaces, Frechet spaces, completely regular spaces, normal spaces, and locally compact spaces.

In general, the terminology of this paper will coincide with the teminology found in [4]. In the comparison of topologies, a topology $\mathscr{T}$ will be weaker than a topology $\mathscr{T}'$ if $\mathscr{T}$ is a subfamily of $\mathscr{T}'$. In this paper, the notation $\mathscr{T}$ denotes the family of all closed sets on X.

DEFINITION 1. A topology $\mathscr{T}$ on a set X is said to be weaker than a topology $\mathscr{T}'$ on X, if, for each closed set F in $(X, \mathscr{T})$, F is also a closed set in $(X, \mathscr{T}')$.

§1. Characterizations of minimal T_0 space.

1.1. DEFINITION A topological space $(X, \mathscr{T})$ is said to be minimal T_0 space, if $\mathscr{T}$ is T_0 topology and there exists no T_0 topology on X strictly weaker than $\mathscr{T}$.

1.2. LEMMA *Let $(X, \mathscr{T})$ be T_0 space on an infinite set X. If there exist A and B in $\mathscr{T}$ such that $\bar{a}=A$, $\bar{b}=B$ and $A\cap B=\phi$, then $(X, \mathscr{T})$ is not minimal T_0 space.*

PROOF Suppose $\mathscr{T}$ is minimal T_0 topology, then there exists $\mathscr{T}'$ which is proper subfamily of $\mathscr{T}$ as the following:

$$\mathscr{T}'=\{T\,|\,T=A\cup A_\alpha\ (b\epsilon A_\alpha)\ \text{or}\ T=A_\beta\ (b\epsilon A_\beta),\ A_\alpha,\ A_\beta\epsilon\mathscr{T}\}.$$

Now, we prove for $\mathscr{T}'$ to be a topology as the following:

$$\Big(\bigcup_{\alpha=1}^{n}(A\cup A_\alpha)\Big)\cup\Big(\bigcup_{\beta=1}^{m}A_\beta\Big)=\Big(A\cup\Big(\bigcup_{\alpha=1}^{n}A_\alpha\Big)\Big)\cup\Big(\bigcup_{\beta=1}^{m}A_\beta\Big)$$

$$=A\cup\Big(\Big(\bigcup_{\alpha=1}^{n}A_n\Big)\cup\Big(\bigcup_{\beta=1}^{m}A_\beta\Big)\Big),$$

where

$$\Big(\bigcup_{\alpha=1}^{n} A_\alpha\Big)\cup\Big(\bigcup_{\beta=1}^{m} A_\beta\Big)\in\mathscr{T},\quad b\in\Big(\bigcup_{\alpha=1}^{n} A_\alpha\Big)\cup\Big(\bigcup_{\beta=1}^{m} A_\beta\Big)$$

Hence, the finite union of closed sets in $\mathscr{T}$ is contained also in $\mathscr{T}'$. And $(\bigcap_{\alpha\in\varDelta}(A\cup A_\alpha))\cap(\bigcap_{\beta\in\varDelta}A_\beta)$ doesn't contain b, so that itself is contained in $\mathscr{T}'$. Here $\mathscr{T}'$ contains ϕ and X clearly.

Next, we prove for $\mathscr{T}'$ to be T_0 topology as the following.

We consider the three cases i.e. (1) the case of $\bar{x}\not\ni b$, $\bar{y}\not\ni b$, (2) the case of $\bar{x}\not\ni b$, $\bar{y}\ni b$, and (3) the case of $\bar{x}\ni b$, $\bar{y}\ni b$, for arbitrary elements x, y in $(X, \mathscr{T}')$. Here, $\bar{x}$ denotes the closure of x in $\mathscr{T}$, and $\bar{x}'$ denotes the closure of x in $\mathscr{T}'$.

1.3. LEMMA *Let $(X, \mathscr{T})$ be a T_0 space on infinite set X. If there exist A and B in $(X, \mathscr{T})$ such that $\bar{a}=A$, $\bar{b}=B$ and $A\cap B=C$, where $C\neq\phi$, $C\neq A$ and $C\neq B$, then $(X, \mathscr{T})$ is not minimal T_0 space.*

PROOF Suppose $\mathscr{T}$ is minimal T_0 topology, then there exists $\mathscr{T}'$ which is proper subfamily of $\mathscr{T}$ as the following:

$\mathscr{T}'=\{T\mid T=A\cup A_\alpha\ (b\in A_\alpha)$ or $T=A_\beta\ (b\notin A_\beta),\ A_\alpha,\ A_\beta\in\mathscr{T}\}$.

Therefore, as well as the method in proof of 1.2 Lemma, we can see easily that $\mathscr{T}'$ is a T_0 topology.

1.4. THEOREM *Let $\mathscr{T}$ be a minimal T_0 topology on an infinite set X: $\mathscr{T}=\{a_\alpha : \alpha\in\varDelta\}$, then arbitrary two elements A_α, A_β in $\mathscr{T}$ have the following relations*

$$A_\alpha\supset A_\beta \quad \text{or} \quad A_\alpha\subset A_\beta.$$

PROOF. Suppose that above result does not happen from the assumption, then it arises following two cases, but the all cases come to have contradiction.

Now, we consider the case of $A_\alpha\cap A_\beta=\phi$. There exist a, b; $a\in A_\alpha$, $b\in A_\beta$, and we have $\bar{a}\cap\bar{b}=\phi$. Hence $(X. \mathscr{T})$s not minimal T_0 space by 1.2 Lemma.

Next, we consider the case of $A_\alpha\cap A_\beta=A$, where A is different from ϕ, A_α and A_β. There exist a and b;

$$\bar{a}\cap\bar{b}\neq\phi \text{ or } \bar{a}\cap\bar{b}=\phi.$$

where

$$a\in A_\alpha-A \quad\text{and}\quad b\in A_\beta-A.$$

Hence $(X, \mathscr{T})$ is not minimal T_0 space by 1.3. Lemma or 1.2. Lemma respectively.

1.5. THEOREM *A necessary and sufficient condition that a T_0 space $(X, \mathscr{T})$ be minimal is that $\mathscr{T}$ satisfies the following properties* (i), (ii).

(i) *Every two closed sets in $\mathscr{T}=\{A_\alpha : \alpha\in\Delta\}$ have the relation $A_\alpha\subset A_\beta$ or $A_\alpha\supset A_\beta$.*

(ii) *If we put $A_\alpha\neq X$ and $A_\alpha\neq\phi$, then A_α is represented by $A_\alpha=\bar{a}_\alpha$ or $A_\alpha=\bigcap_{\beta\in\Delta}\bar{a}_\beta$.*

PROOF At first we will show that the conditions (i), (ii) are necessary. We put

$$\mathscr{T}'=\mathscr{T}-\{A_\alpha : \alpha\in\Delta',\ \Delta'\subsetneqq\Delta\},$$

and we show as following, that $(X, \mathscr{T}')$ is not T_0 space;

Now we consider the case of $A_\alpha=\bar{a}_\alpha(\alpha\in\Delta')$ in $\mathscr{T}$.

Let $\bar{a}'_\alpha=A_\beta$ in $\mathscr{T}'$, then $A_\alpha\subset A_\beta$. If $A_\beta=\bar{a}_\beta$ in $\mathscr{T}$, then $A_\beta=\bar{a}'_\beta$. Hence $\bar{a}'_\alpha$ is equal to $\bar{a}'_\beta$ in $\mathscr{T}'$, which is contrary. And if A_β is not a point closure in $\mathscr{T}$, then $A_\alpha\neq A_\beta$ and $A_\alpha\subset A_\beta$. Hence there is an element x such that $x\in A_\beta-A_\alpha$. Since $\bar{a}_\alpha'=A_\beta$, A_β is the least closed set containing a_α in $\mathscr{T}'$. Since $\bar{x}'\subseteq A_\beta$, let $\bar{x}' =A_\lambda$, then $A_\lambda\subseteq A_\beta$. On the otherhand, since $\bar{x}\supset A_\alpha$, we have $a_\alpha\in A_\lambda$, $\bar{a}'_\alpha\subseteq A_\lambda$ and $A_\beta\subseteq A_\lambda$. Hence we have $A_\lambda=A_\beta$ and $\bar{x}'=\bar{a}'_\alpha$, which is contrary.

And, we consider the case of $A_\alpha=\bigcap_{\beta\in\Delta}\bar{a}_\beta$ in $\mathscr{T}$.

Here, we can see $A_\alpha\in\mathscr{T}$ and $A_\alpha \bar{\in} \mathscr{T}'$. On the otherhand, since $\bar{a}_\beta \in\mathscr{T}'$, we have $\bigcap_{\beta\in\Delta}\bar{a}_\beta=A_\alpha \in \mathscr{T}'$.

Therefore ths case can't arise.

Next, we show that the conditions (i), (ii) are sufficient.

Let $\mathscr{T}$ be a minimal T_0 topology which does not satisfies (i) or (ii). But by 1.4. theorem, we know that $\mathscr{T}$ satisfies (i). Hence, this assumption represents that there exists an element A_β of $\mathscr{T}$ which doesn't satisfy (ii) only. We put.

$$\mathscr{T}'=\mathscr{T}-A_\beta$$

then clearly $\mathscr{T}'$ is a topology and $(X, \mathscr{T}')$ is T_0 space.

It is contrary.

1.6. REMARK As following example, we know that $(X, \mathscr{T})$ is not minimal T_0 space, if $\mathscr{T}$ is a family of non-linearly ordered sets. Now, let X be set $\{a, b, c\}$ and $\mathscr{T}$ be a closed family in X; ϕ, $\{c\}$, $\{b, c\}$, $\{a, c\}$, X, then $\mathscr{T}$ is a topology clearly. And point closures of each elements are different mutually, since $\bar{a}=\{a, c\}$, $\bar{b}=\{b, c\}$, $\bar{c}=\{c\}$. Hence $(X, \mathscr{T})$ is T_0 space, but $\mathscr{T}$ is a family of non-linearly ordered sets. Here we put that $\mathscr{T}'$ is a family of closed sets excepting $\{a, c\}$ in $\mathscr{T}$, then $\mathscr{T}'$ is also T_0 topology and $\mathscr{T}'$ is strictly weaker than $\mathscr{T}$.

Finally $(X, \mathscr{T})$ is not minimal T_0 space.

1.7. REMARK On 1.5. theorem, the condition (i) only is not sufficient for T_0 space to be minimal. As an example, we put

$$X=\{[1, 2], 3, 4, 5, 6\cdots\},$$

and $\mathscr{T}$ is a family of closed sets in X such that

$$X,\ \{[1, 2], 3, 4, 5, \cdots\},\quad \{[a_i, 2], 3, 4, 5, \cdots\},$$
$$\{2, 3, 4, 5, \cdots\},\quad \{3, 4, 5, \cdots\},\ \cdots,$$

where a_i is a real number such that $1<a_i<2$.

Then $\mathscr{T}$ is a family of linearly ordered sets. Now $(X, \mathscr{T})$ is T_0 space clearly, but we put that $\mathscr{T}'$ is a family of closed sets excepting $\{[1, 2], 3, 4, 5, \cdots\}$ in $\mathscr{T}$, then $\mathscr{T}'$ is also T_0 topology and $\mathscr{T}'$ is strictly weaker than $\mathscr{T}$. Finally, $(X, \mathscr{T})$ is not minimal T_0 space.

§ 2. Characterizations of minimal T_D space.

2.1. DEFINITION A topological space $(X, \mathscr{T})$ is said to be minimal T_D-space, if $\mathscr{T}$ is T_D topology and there exists no T_D topology on X strictly weaker than $\mathscr{T}$.

As the ground of this part, we begin with the following theorem in paper [1].

2.2. THEOREM *Let $\mathscr{T}$ be a topology in a set X.*

(1) *If $(X, \mathscr{T})$ is a T_1 space, then $(X, \mathscr{T})$ is T_D space.*

(2) *If $(X, \mathscr{T})$ is a T_D space, then $(X, \mathscr{T})$ is T_0 space.*

2.3. THEOREM *Let $\mathscr{T}$ be a minimal T_D topology on an infinite set X; $\mathscr{T}=\{A_\alpha : \alpha\epsilon\Delta\}$, then arbitrary elements A_α, A_β in $\mathscr{T}$ have following relations*

$$A_\alpha \supset A_\beta \quad or \quad A_\alpha \subset A_\beta.$$

We can easily see above theorem by following 2.4. Lemma and 2.5. Lemma.

2.4. LEMMA *Let $(X, \mathscr{T})$ be T_D space on an infinite set X. If there exist A and B in $(X, \mathscr{T})$ such that $\bar{a}=A$, $\bar{b}=B$ and $A\cap B=\phi$, then $(X, \mathscr{T})$ is not minimal T_D space.*

2.5. LEMMA *Let $(X, \mathscr{T})$ be a T_D space on infinite set X. If there exist A and B in $(X, \mathscr{T})$ such that $\bar{a}=A$, $\bar{b}=B$ and $A\cap B=C$, where $C\neq\phi$, $C\neq A$ and $C\neq B$, then $(X, \mathscr{T})$ is not minimal T_D space.*

We can see easily above Lemma 2.4 and 2.5 in the similar method with the proof of 1.3 and 1.4.

2.6. THEOREM *A necessary and sufficient condition that a T_D space $(X, \mathscr{T})$ be minimal is that $\mathscr{T}$ satisfies following properties* (i), (ii).

(i) *Every two closed sets in $\mathscr{T}=\{A\alpha : \alpha\epsilon\Delta\}$ has relation*

$$A_\alpha \subset A_\beta \quad or \quad A_\alpha \supset A_\beta.$$

(ii) *If we put $A_\alpha \neq X$ and $A_\alpha\neq\phi$, then A_α satisfies*

$$A_\alpha=\bar{a}_\alpha,\ A_\alpha=\bar{a}_\alpha-a_\alpha \ or \ A_\alpha=(\bigcap_{\alpha\epsilon\Delta}\bar{a}_\alpha)\cap(\bigcap_{\beta\epsilon\Delta}(\bar{a}_\beta-a_\beta)).$$

PROOF At first we will show that the conditions (i), (ii) are necessary. We put

$$\mathscr{T}'=\mathscr{T}-\{A_\beta : \beta\epsilon\Delta',\ \Delta'\subsetneqq\Delta\},$$

and we show as following, that $(X, \mathscr{T}')$ is not T_D space;

Now we consider the case of $A_\beta=\bar{a}_\beta$ $(\beta\epsilon\Delta')$ in $\mathscr{T}$. Let $\bar{a}_\beta{}'=A_\lambda$ in $\mathscr{T}'$ then $A_\beta\subset A_\lambda$. If $A_\lambda=\bar{a}_\lambda$ in $\mathscr{T}$ then $A_\lambda=\bar{a}'_\lambda$ in $\mathscr{T}'$.

Hence $\bar{a}'_\beta$ is equal to $\bar{a}'_\lambda$ in $\mathscr{T}'$, which is contrary. Therefore, we can see that $(X, \mathscr{T}')$ is not T_0 space, i.e., $(X, \mathscr{T}')$ is not T_D space. And if $A_\lambda=\bar{a}_\lambda-a_\lambda$ in $\mathscr{T}$, then $A_\lambda\neq A_\beta$ and $A_\lambda\supset A_\beta$. Hence there exists an element x such that $x\epsilon A_\lambda-A_\beta$ and $x\neq a_\beta$. Since we put $\bar{x}'=A_\mu$, then $A_\mu\subseteq A_\lambda$. On the otherhand, a_β

is an element of A_μ. Hence we have $\bar{a}'_\beta \subseteq A_\mu$, i.e., $A_\lambda \subseteq A_\mu$. Therefore we may have $A_\lambda = A_\mu$, i.e. $\bar{a}'_\beta = \bar{x}'$. We can see that $(X, \mathscr{T}')$ is not T_0 space, i.e., $(X, \mathscr{T}')$ is not T_D space.

And we consider the case $A_\beta = \bar{a}_\beta - a_\beta (\beta \in \Delta')$ in $\mathscr{T}$. From the first case, we can see that all $\bar{a}_\beta$ contain $\mathscr{T}'$. Since $\mathscr{T}'$ is T_D topology, all $\bar{a}_\beta - a_\beta$ are the elements of $\mathscr{T}'$. Hence this case does not arise.

Next we consider the case $A_\beta = (\bigcap_{\alpha \in \Delta} \bar{a}_\alpha) \cap (\bigcap_{\gamma \in \Delta} (\bar{a}_\gamma - a_\gamma))$ $(\beta \in \Delta')$ in $\mathscr{T}$. Here $\bigcap_{\alpha \in \Delta} \bar{a}_\alpha$ does not fall away in $\mathscr{T}'$ under the first case, and similarly for $\bigcap_{\gamma \in \Delta} (\bar{a}_\gamma - a_\gamma)$ under the second case. Therefore, A_β must be an element of $\mathscr{T}'$. It is contrary.

The proof of the sufficiency is the similar method in proof of 1.5. Theorem.

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