A PROBLEM OF CONNECTED-OPEN TOPOLOGY FOR FUNCTION SPACE

By Sang Moon Kim

0. Introduction

In the paper [1] Irudayanatan, Anthony and Somashekar Naimpally considered a function space topology T for Y^X in which the function space of connected functions is closed when Y is completely normal. He named the topology as connected open topolgy.

He proved following theorems in his paper;

0.1. *If Y is completely normal, then C^{-2} is closed in (Y^X, T).*

0.2. *If Y is locally connected then the compact open topology or k-topology for Y^X is smaller than T.* (see page 221 of [2])

Here T denotes the connected open topology for Y^X, and in theorem 0.1 the space Y^X has connected open topology.

He posed following problems;

0.3. *Is theorem* 0.1 *ture if Y is not completely normal?*

0.4. *If Y is not locally connected, is theorem* 0.2. *ture?*

We shall show examples which shows that if Y is not ompletely normal then (theorem 0.1) is not true and if Y is not locally connected then (theorem 0.1.) is also not true.

1. Definitions and Examples.

1.1. DEFINITION : A function $f: X \to Y$ is connected function if and only if for each connected set $K \subset X, f(K)$ is connected subset of Y.

1.2. DEFINITION: For each connected subset K of X and each pair of open subsets U, V of Y, let $W(K; U, V) = \{f \epsilon Y^X : f(K) \subset U \cup V,\ f(K) \cap U = f(K) \cap V \neq \phi\}$. The topology T generated by the subbasis $\{W(K: U, V)\}$ is called the connected-open topology for Y^X.

C^{-2} denotes the subset of Y^X consisting of all connected functions.

In paper (3) William J. Pervin made an example which shows the(proposition 0.3) is not true. We make also an example which shows she (proposition 0.3) is not true.

1.3. EXAMPLE; Let X and Y be the sets of all integers. Let $\mathscr{T}$ be the minimal

T_1 topology for X and Y, that is $\mathscr{F}=\{X-A : A=\textit{finite subset of } X\}$.

We define a function$f$: $\begin{cases} f(x)=1, & \text{if } x \text{ is an odd number} \\ f(x)=2, & \text{if } x \text{ is an even number.} \end{cases}$

Then f is a function from X to Y and since in the topological space Y a suset K is connected if and only if K is an infinite subset or an one element subset of X, where f maps the whole space X on to the disconnected set $\{1,2\}$ hence f is not a connected function.

Now let T be the connected open topology for Y^X. We consider a neighborhood of the function f.

Let $S=\bigcap_{i=1}^{N} W^i(K^i : U^i, V^i)$ be a subbase element of T containing the function f. We divide the set $\{K^i : i=1,2,\cdots N,\}$ into three sets $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{A}_3$ as following.

$\mathfrak{A}_1=\{K^i : K^i$ contains even and odd numbers $i=1,2,\cdots,m\}$

$\mathfrak{A}_2=\{K^i : K'^i$ contains only odd numbers $i=1,2,\cdots,n\}$

$\mathfrak{A}_3=\{K''^i : K^i$contains only even numbers $i=1,2,\cdots,p\}$

By same way as above we rename U's and V's into three groups as U^i, U'^i and U''^i ; V^i, V'^i and V''^i. Then we can write S as following:

$$S=(\bigcap_{i=1}^{m} W^i(K^i;U^i,V))\cap(\bigcap_{i=1}^{n} W^i(K'^i;U'^i,V'^i))\cap(\bigcap_{i=1}^{p} W^i(K^i;U''^i,V''^i)).$$

We know $f(K^i)=\{1,2\}, f(K'^i)=\{1\}$ and $f(K''^i)=\{2\}$.

Hence we can assume $1\in U^i, 2\in V^i, 1\in U'^i, 1\in V'^i, 2\in U''^i$ *and* $2\in V''^i$.

Let $U=(\bigcap_{i=1}^{m} U^i)\cap(\bigcap_{i=1}^{n} U'^i)\cap(\bigcap_{i=1}^{n} V'^i)$ and $V=(\bigcap_{i=1}^{m} V^i)\cap(\bigcap_{i=1}^{p} V''^i)\cap(\bigcap_{i=1}^{p} U''^i)$, then we have $1\in U$ and $2\in V$.

We linearly order U and V as following:

$U=\{1, a_1, a_2, a_3, \cdots\cdots\cdots\}$

$V=\{2, b_1, b_2, b_3, \cdots\cdots\cdots\}$.

We consider three cases about $\mathfrak{A}_i$:

(1) $\mathfrak{A}_1$ is non empty, or $\mathfrak{A}_x$ is empty but $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are non empty then in this case both U and V are non empty sets.

(2) Both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are empty set but $\mathfrak{A}_3$ is non empty set then U is empty but V is non empty.

(3) Both $\mathfrak{A}_1$ and $\mathfrak{A}_3$ are empty sets but $\mathfrak{A}_2$ is non empty then U is non empty but V is an empty set.

We define three connected function following above three cases

In case (1) define g_1 :

$$\begin{cases} g_1(\pm1)=1, g_1(\pm3)=a_1, g_1(\pm5)=a_2, \cdots\cdots \\ g_1(0)=g_1(\pm2)=2, g_1(\pm4)=b_1, g_1(\pm6)=b_2, \cdots\cdots \end{cases}$$

In case (2) define g_2:

$$\begin{cases} g_2(x)=x, \text{ if } x \text{ is an odd number,} \\ g_2(0)=g_2(\pm2)=2, g_2(\pm4)=b_1, \cdots\cdots \end{cases}$$

In case (3) define g_3:

$$\begin{cases} g_3(\pm1)=1, g_3(\pm3)=a_1, g_3(\pm5)=a_2, \cdots\cdots \\ g_3(x)=x, \text{ If } x \text{ is an even number.} \end{cases}$$

Then clearly g_1, g_2 and g_3 are connected function from X to Y, and in each case S contains g_i respectively.

Hence any neighborhood of S contains a connected function and hence C^{-2} is not closed in (Y^X, T).

By these we made the example we wanted. and proved that the proposition 0.3 is not ture.

Now we shall prove the proposition 0.4 is not true if X is not a locally compact.

1.4 *Let Y be the real numbers with usual topology and X be all rational numbers contained in R, and consider X as a subspace of the space R. Then the compact open topology of Y^X is exactly bigger than the connected-open topology of Y^X*

PROOF Let T be the connected-open topology of Y^X then T has a subbase consisting of

$$W(K;U,V)=\{f: f(K)\subset U\cup V, f(K)\cap U=f(K)\cap V\neq\phi\}.$$

The compact open topology is defined as following: For each subset K of X and each subset U of Y, define $W(K,U)$ to be the set of all members of Y^X which carry K into U: that is, $W(K,U)=\{f; f(K)\subset U\}$. The family of all sets of the form $W(K,U)$, for each compact subset K of X and U open in Y, is a sub-base for the compact open topology for Y^X.

But we know that X is extremely disconnected space hence the only connected components of X are the one point subset of X.

Hence $W(K; U,V)$ the subbase element of compact-open topology has simple form $W(x;U,U)=\{f\in Y^X : f(x)\in U\}$.

In this case the connected open topology coincides with the product topology of Y^X.

Where the subbasis of compact open topology of Y^X consists of sets of the form:

$W(K,U)=\{f: f(K)\subset U\}$, K is a compact subset of X and U is an open subset of R.

Let $K=\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, 0\right\}$ then K is compact subset of X.

Let $U=(1,2)=\{x : 1<x<2\}$ then U is open in R. We easily see that (K,U) is not open in connected open topology of Y^X.

Since (x,U) is clearly an open set of compact open topology, compact open topology is exactly bigger than the conneced open topology of Y^X. This example shows that the statement 0.4 is not true.

As a by product we have following theorem:

1.5. *X is an extremely disconnected space and Y is a topological space, then the product topology coincides with the connected-open topology for Y^X, and both are not bigger than the compact open topolgy of Y^X.*

1.6. *Let R be the real numbers with usual topology and let X be the rational subspace of R, then the product topology of Y^X and the connected open topology of Y^X are identical, and both are exactly smaller than the compact open topology.*

NOTE In (1) the author considered the connected-open topology to obtain the least topology in which the set of connected functions are closed in the function space and he give the condition of completely normal of the range space, now by proposition 1.6 we know that the character of domain space has also a great relation to the character whether the set of connected function is closed or not.

Mathematics Department
Liberal Arts and Science College
Kyungpook University.

REFERENCES

[1] Irudayanathan, Anthony and Somashekhar, Naimpally: *Connected-open topology for function spases, Indag. Math., Vol.* 28, *Fasc.* 1. (1966)

[2] Kelly, J. L.: *General Topology,* Van Nostrand Company, (1955)

[3] Pervin, William, J.: *On the Connected-open topology.* Indag. Math., Vol. 28, Fasc. 1 (1967).

[4] Birkhoff, G.: Lattice theory. (1948)