Strong Condition that a Stationary Stochastic Process is Ergodic

Yeong-Don Kim

1. **Introduction** Let us assume that $X(t)$ is the stochastic process defined in regard to t in $T=\{0, \pm1, \cdots\}$ by J. L. Doob. [1, p. 49] Let us also assume that $X(t)$ is defined in probability space $(\Omega, \mathcal{A}, P)$ in which Ω is an abstract set, $\mathcal{A}$ is a σ-field of subset of Ω, and P is a probability measune in $\mathcal{A}$. Let us again assume that, in $(\Omega, \mathcal{A}, P)$, the expected value $E(X)$ in regard to the random variable Y is defined if $EX=\int_\Omega Y\, dP$ exists.

A stochastic process is defined to be strictly stationary if for any integer k and any choice of k points $t_1 \cdots, t_k$ in T, the joint distribution of the random variables $X(t_1+h), \cdots, X(t_k+h)$ does not depend on h. For a strictly stationany stochastic process one may establish the following law of large numbers; for any Borel function $g(X_1, \cdots, X_k)$ of k variables such that the ensemble mean

$$E|g| = E|g\{X(t_1+h), \cdots, X(t_k+h)| < \infty,$$

the sample means

$$M_n(g) = \frac{1}{n+1}\sum_{h=0}^{n} g\{X(t_1+h), \cdots, X(t_k+h)\}$$

converge with probability one to a random variable, donoted by $\bar{g}$. Since the process is transitive, the limit $\bar{g}$ is identically constant with probaility one, and is equal to the ensemble mean

$$E(g) = Eg\{X(t_1+h), \cdots, X(t_k+h)\}.$$

In case the limit $\bar{g}$ and $E(g)$ converge, the application of the theory of a stochastic process may be interest, and worthy of considerations.

A strictly stationary stochastic process $X(t)$ which is strongly engodic will be defined in relation to all Borel function $g(X_1, \cdots, X_k)$ with k variables (k: positive integer), the sample mean $M_n(g)$ conveging to $E(g)$.

Proving a fact that a strictly stationary stochastic process is metrically transitive if it is strongly ergodic may be considered very interesting.

In this case, we give necessary and sufficient conditions in terms of the characteristic functions of the process that it be strongly ergodic. It is apparent that random variables $X(t_1+h), \cdots, X(t_k+h)$ which form a joint distribution will be defined, regardless of the value of h, as a strictly stationary stochastic process of order k in relation to all k points $t_1, \cdots, t_k$, obtained arbitrily from T.

A stochastic process with the mean ergodic of order k(k: positive integer) will be defined in relation to all k variables Borel function $g\ (X_1, \cdots, X_k)$ in which

$$E|g|^r = E|g\{X(t_1+h), \cdots, X(t_k+h)\}|^r < \infty, \quad (1 \leq r \leq 2).$$

The sample means $M_n(g)$ converges to $E(g)$ in r-mean. In other words, $E|M_n(g) - E(g)|^r \longrightarrow 0$ when $n \longrightarrow \infty$. Similarly a process which is strongly ergodic of order k is defined if $M_n(g)$ converges to $E(g)$ as a probarbility value.

The purpose of this paper is to give condititions in terms of characteristic functions that a stochastic process is mean Ergodic of order k.

2. Results the author obtained are the fodlowings:

Theorem 1. Conditions that a stochastic process stationary of order k is ergodic of order k:

Assume that the random variables $X(t)$ is defined in relation to all t within $T = \{0, \pm 1, \pm 2, \cdots\}$, and k is a positive integer, and that $t_1, \cdots, t_k$ are points within T.

In R_k(k-dimensional Euclidean space), a stochastic function is assumed to be:

$$(1)\quad \varphi(u_1, \cdots, u_k) = \int \cdots \int_{R_k} e^{i\left(\sum_{j=1}^{k} u_j X_j\right)} dF(X_1, \cdots, X_k)$$

of course, here in relation to all real numben $u_j (j = 1, 2, \cdots, k)$,

$$(2)\quad E\left\{e^{i[u_1 X(t_1+h) + \cdots + u_k X(t_k+h)]}\right\} = \varphi(u_1, \cdots, u_k), \text{ for all } h \in T.$$

A characteristic function $\varphi(u_1, \cdots, u_k; \tau)$ in relation to all τ belong to T is assumed to be:

(3) $E\{e^{i[u_1\{X(t_1+h)-X(t_1+h+\tau)\}+\cdots+u_k\{X(t_k+h)-X(t_k+h+\tau)\}]}\} = \varphi(u_1, \cdots, u_k; \tau)$

for all $h \in T$.

Let us assume $1 \leq r \leq 2$. Then the sample mean $M_n(g)$ converges as a limit in r-mean in relation to all Borel function $g(X_1, \cdots, X_k)$ with $E|g|^r < \infty$. If the contents of $\varphi(u_1, \cdots, u_k)$ and $\varphi(u_1, \cdots, u_k; \tau)$ are satisfied, the sample mean $M_n(g)$ converges to the ensemble mean $E(g)$ in r-mean if and only if

(4) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{\tau=0}^{n} \varphi(u_1, \cdots, u_k; \tau) = |\varphi(u_1, \cdots, u_k)|^2$

in relation to all real number $u_1, \cdots, u_k$.

Proof. First of all, what is necessary is to prove that given as $g(X_1, \cdots, X_k) = e^{i(u_1X+\cdots+u_kX_k)}$ holds in relation to $u_1, \cdots, u_k$ given as real numbers. Let $Y(t) = X(t) - X(t+\tau)$ be a strictly stationary of order k.

The stochastic process

$$Y(t) = e^{i[u_1X_1(t_1+t)+\cdots+u_kX_k(t_k+t)]} - \varphi(u_1, \cdots, u_k)$$

is a wide-sense stationany stochastic process with zero mean by (2) and (3), and at the same time covariance function

$$R_Y(t) = EY(t)Y^*(t+\tau) = \varphi(u_1, \cdots, u_k; \tau) - |\varphi(u_1, \cdots, u_k)|^2$$

where the asterisk denotes a complex conjugate [2].

In a wide-sense stationary stochastic process,

$$\frac{1}{n+1}\sum_{t=0}^{n} Y(t) = M_n(g) - E(g)$$

converges as a limit in quadradic mean in the law of large numbers.

Furthermore, the limit random variable becomes zero if and only if $\lim_{n\to\infty}\frac{1}{n+1}\sum_{\tau=0}^{n} R_Y(t) = 0$, the equivalent of $\lim_{n\to\infty}\frac{1}{n+1}\sum_{\tau=0}^{n}\varphi(u_1, \cdots, u_k; \tau)$. Thus theorem 1 has been proved for Borel function g which are complex exponentials.

Next, let us assume that g is Borel function with $E|g| < \infty$ and $\varepsilon > 0$.

Then, there is a trigonometric polynomial

$$g_\varepsilon(X_1, \cdots, X_k) = \sum_{j_1, \cdots, j_k=1}^{m} c(j_1, \cdots, j_k) e^{i(u_{j_1}X_1 + \cdots + u_{j_k}X_k)}$$

such that

$$E|g - g_\varepsilon|^r = \int \cdots \int_{R_k} |g(X) - g_\varepsilon(X)|^r dF(X) < \varepsilon^r.$$

First of all, let us prove that the limit of mean $\lim_{n\to\infty} M_n(g)$ exists.

According to Minkowskis inequality

$$E^{\frac{1}{r}}|M_n(g) - M_m(g)|^r \le E^{\frac{1}{r}}|M_n(g) - M_m(g_\varepsilon)|^r + E^{\frac{1}{r}}|M_n(g_\varepsilon) - M_m(g_\varepsilon)|^r$$
$$+ E^{\frac{1}{r}}|M_m(g_\varepsilon) - M_m(g)|^r$$

where n and m are arbitrary. If $n,m \to 0$

$$E^{\frac{1}{r}}|M_n(g_\varepsilon) - M_m(g_\varepsilon)|^r \le E^{\frac{1}{2}}|M_n(g_\varepsilon) - M_m(g_\varepsilon)|^2 \longrightarrow 0,$$

$$E^{\frac{1}{r}}|M_n(g) - M_n(g_\varepsilon)|^r \le \frac{1}{n+1} \sum_{t=0}^{n} E^{\frac{1}{r}}|g(X(t)) - g_\varepsilon(X(t))|^r$$
$$= E^{\frac{1}{r}}|g - g_\varepsilon|^r < \varepsilon.$$

In other words, if

$$\lim_{n,m\to 0} E^{\frac{1}{r}}|M_n(g) - M_m(g)|^r \le 2\varepsilon$$

under the condition $\varepsilon \to 0$, it follows that $\lim_n M_n(g)$ exist as a limit in r-mean.

Similary,

$$E^{\frac{1}{r}}|M_n(g) - Eg| \le E^{\frac{1}{r}}|M_n(g) - M_n(g_\varepsilon)|^r + E^{\frac{1}{r}}|M_n(g_\varepsilon) - Eg_\varepsilon|$$
$$+ |Eg_\varepsilon - Eg|. \text{ Q.E.D}$$

Corollary 1. A stochastic process satisfying (1), (3) and (4) is mean ergodic of k.

Together with Birkhott-Khintchine ergodic theorem we immediately obtain the following theorem.

Theorem 2. Conditions that a strictly stationary stochastic process be strongly ergodic;

A strictly stationary stochastic process $X(t)$, which is defined in relation to t within $T = \{0, \pm 1, \cdots\}$, is strongly ergodic if and only if arbitrary real numbens $u_1, \cdots, u_k$ satisfy (4) in $\varphi(u_1, \cdots, u_k)$ and $\varphi(u_1, \cdots, u_k;\ \tau)$ which are defined respectively in (2) and (3). It is sometimes of interest to consider a sequence of sample mean of the form

$$(5)\quad M_n(g_n, t) = \frac{1}{n+1}\sum_{t=0}^{n} g_{n,t}(X(t_1+t), \cdots, X(t_k+t))$$

where the Borel function $g_{n,t}(X_1, \cdots, X_k)$ converge in some sense, as $n\to\infty$ to a function $g(X_1, \cdots, X_k)$.

Theorem 3. Let (2) and (3) hold. let $1\leq r\leq 2$, and let $g(X_1, \cdots, X_k)$ and $g_{n,t}(X_1, \cdots, X_k)$ for $n=0, 1, 2, \cdots$, and $t=0, 1, 2, \cdots, n$, be Borel function such that $E|g|^r<\infty$, $E|g_{n,t}|^r<\infty$,

and

$$(6)\quad \lim_{n\to\infty}\frac{1}{n+1}\sum_{t=0}^{n} E^{\frac{1}{r}}|g_{n,t} - g|^r = 0.$$

Then the sample means $M_n(g_{n,t})$, defined by (5), converge in r-mean. Further, if (4) holds, then they converge in r-mean to $E(g)$.

Proof: The theorem follows immediately from theorem 1 and the inequality

$$E^{\frac{1}{r}}|M_n(g_{n,t}) - M_n(g)|^r \leq \frac{1}{n+1}\sum_{t=0}^{n} E^{\frac{1}{r}}|g_{n,t} - g|^r.$$

As an instance of a case where this theorem is required, consider the sample correlation function defined by, for $0\leq v<n$,

$$R_n(v) = \frac{1}{n}\sum_{t=0}^{n-v} X(t)\cdot X(t+v)$$

which may be written in the form of (5), with $t_1=0$ and $t_2=v$, if one defines

$$g_{n,t}(X_1, X_2) = \begin{cases} X_1X_2 & \text{for } t=0, 1, \cdots, n-v. \\ 0 & \text{for } t=n-v+1, \cdots, n. \end{cases}$$

It seems disirable to investigate conditions under which the sample means of a stochastic process which is strictly stationary of order k converge strongly, that is, with probability one. By the method of proof of theorem 1 we have not been able to obtain a theorem stating conditions under which, for any Borel function g, the sample mean $M_n(g)$ converge to $E(g)$ with probability one.

However, we state such a theorem for bounded Borel function g whose points of discontinuity form a set of F-measure 0 (when F-measure is $\dot{R}_k$ by the distribution F given in (1)).

Theorem 4. Conditions that a stochastic process stationary of order k be partially strongly of order k:

Let (2) and (3) hold. Suppose that for all real $u_1, \cdots, u_k$ there exist positive constant k_0 and α such that, for every n,

$$(7)\quad \frac{1}{n+1}\sum_{t=-n}^{n}\left(1-\frac{|\tau|}{n+1}\right)(\varphi(u_1,\cdots,u_k;\tau)-|\varphi(u_1,\cdots,u_k)|^2)\leqq\frac{k_0}{n^\alpha}.$$

Then for every bounded Borel function $g(X_1, \cdots, X_k)$ whose points of discontinuity in R_k form a set of F-measure the sample means $M_n(g)$ converge to $E(g)$ with probability one.

All afore-mentioned considerations, may be, without exception, expanded to the case of continuous parameter stochastic processes. They are expanded to some cases in which a concept of stationary stochastic process is defined, and the law of large numbers is recognized in relation to a wide-sense stationbry stochastic processes. If we are to stress this point, we will see that the following theorem hold true.

Theorem 5. Conditions that a continuous parameter stochastic process stationary of order k be mean ergodic of order k:

Assume that random variables $X(t)$ has been defined in relation to t in $T=\{-\infty<t<\infty\}$, and $t_1, \cdots, t_k$ are points in T. Let us also assume that characteristic function $\varphi(u_1, \cdots, u_k)$, connected by corresponding distribution function $F(X_1, \cdots, X_k)$, satisfy

$$Ee^{i\{(u_1X(t_1+h)+\cdots+u_kX(t_k+h))\}}=\varphi(u_1, \cdots, u_k) \text{ for all } h\in T.$$

Let us further assume that a characteristic function $\varphi(u_1, \cdots, u_k;\tau)$, which satisfies (3) in relation to each τ bolong to T, exist. Again, let us assume that $\varphi(u;\tau)$ is continuous for τ at $\tau=0$ in relation to all $u=(u_1, \cdots, u_k)$, $(r; 1\leq r\leq 2)$.

In relation to a all Borel funcction $g(X_1, \cdots, X_k)$, which is

$$(8)\quad E|g|^r=\int_{R_k}|g|^r dF<\infty$$

the sample mean

(9) $M_T(g) = \frac{1}{T}\int_0^T g(X(t_1 + t)\cdots, X(t_k + t))dt,$

exist, and it converges as a limit in r-mean.

If, (2) and (3) being satisfied, the sample mean $M_T(h)$ converges in r-mean to $E(g)$ only when

(10) $\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(u;\tau)d\tau = |\varphi(u)|^2$

in relation to all $u = (u_1, \cdots, u_k)$.

3. **Conclusions** The author intends to draw a conclusion citing an example in which the afore-mentioned items are applied to the stationary analysis of time series.

Let us assume that $X(t)$ is the sequence of random variables of determined distribution. It does not have to be independent.

$F(X)$, $\varphi(u)$, m and σ^2 mean distribution, characteristic function, mean, and finite variance.

The sample distribution function

$$F_n(X) = \frac{1}{n}\sum_{t=1}^{n} W_X(X(t)); \quad W_x(y) = \begin{cases} 1 & (y < x) \\ 0 & (y \geq x) \end{cases}$$

The sample characteristic function

$$\varphi_n(u) = \frac{1}{n}\sum_{t=0}^{n} e^{iuX(t)}.$$

The sample mean

$$M_n = \frac{1}{n}\sum_{t=0}^{n} X(t),$$

and the sample variance

$$\sigma^2 = \frac{1}{n}\sum_{t=0}^{n} X^2(t) - M_n^2.$$

In afore-mentioned theorems, we see that the volumes will agree.

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Sungkyunkwan University