Some properties of an operator set $A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X]$

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1. Preliminaries. It is well known that the set $B(X, X)$ of all bounded linear operators on Banach space X into Banach space X is also a Banach space. If X is a Banach space, then $Y=X\times X\times\cdots\times X$ is a Banach space with the norm $\|(x_1, x_2, \cdots, x_N)\| = \sup_{1\leq k\leq N} \|x_k\|$. Let $T_k \in B(X, X)$, $k=1, 2, \cdots N$, we set the domain $D(\xi I-T)=D_T(\xi)$ and Range $R(\xi I - T) = \Delta_T(\xi)$ for any $\xi \in \rho(T)$ where $\rho(T)$ is a resolvent set of the operator T. Then we have $D(R_\xi(T)) = \Delta_T(\xi)$ and $R(R_\xi(T)) = D_T(\xi)$.

Through this paper $R_\xi(T)$, $\sigma(T)$ and I will denote a resolvent operator, spectrum and identity transformation respectively. The resolvent operator can be regarded as a bounded linear operator on $\Delta_T(\xi)$ onto $D_T(\xi)$. Defining

$$R_{\xi_1}(T_1)R_{\xi_2}(T_2)\cdots R_{\xi_N}(T_N)(x_1, x_2, \cdots, x_N) = [R_{\xi_1}(T_1)x_1][R_{\xi_2}(T_2)x_2]\cdots\cdots[R_{\xi_N}(T_N)x_N]$$

for $(x_1, x_2, \cdots, x_N) \in \prod_{k=1}^{N} \Delta_{Tk}(\xi_k)$, we have

$$f(T_1, T_2, \cdots, T_N)(x_1, x_2, \cdots, x_N) = \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} f(\xi_1, \xi_2, \cdots, \xi_N)[R_{\xi_1}(T_1)X_1]\cdots[R_{\xi_N}(T_N)X_N]d\xi_1\cdots d\xi_N.$$

An operator $f(T_1, T_2, \cdots, T_N): \prod_{k=1}^{N} \Delta_{Tk}(\xi_k) \longrightarrow X$ is a multilinear transformation. We denote this by

$$A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X] = \{f(T_1, T_2, \cdots, T_N): \prod_{k=1}^{N} \Delta_{T_k}(\xi_k) \longrightarrow X\}.$$

Defining $\|f(T_1, T_2, \cdots, T_N)\| = \sup_{\|(X_1, X_2, \cdots, X_N)\|\leq 1} \|f(T_1, T_2, \cdots, T_N)\ (x_1, x_2, \cdots, x_N)\|$, it enjoys following properties:

1) $\|f(X_1, X_2, \cdots, X_N)\| \geq 0$

2) $\|f(T_1, T_2, \cdots, T_N) + g(T_1, T_2, \cdots, T_N)\| \leq \|f(T_1, T_2, \cdots, T_N)\| + \|g(T_1, T_2, \cdots T_N)\|$

3) $\|\alpha f(T_1, T_2, \cdots, T_N)\| = |\alpha|\ \|f(T_1, T_2, \cdots, T_N)\|$

Definition 1. By the strong opeorator topology in the set $A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X]$ we mean a topology induced by the norm $\|\cdot\|$ which is defined in $A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X]$.

A spherical neighborhood is given by

$$S_\varepsilon(f(T_1, T_2, \cdots, T_N): (A_1 \times A_2 \times \cdots \times A_N))$$
$$= \{g(T_1, T_2, \cdots, T_N) \in A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X]: \| f(T_1, T_2, \cdots, T_N)$$
$$- g(T_1, T_2, \cdots, T_N)(x_1, x_2, \cdots, x_N) \| < \varepsilon, \quad (x_1, x_2, \cdots, x_N) \in \prod_{k=1}^{N} A_k\}$$

where each A_k $(k=1, 2, \cdots, N)$ is any finite set in X. This basis induces a topology in $A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X]$.

In this topology a generalized sequence $\{f_\alpha(T_1, T_2, \cdots, T_N)\}$ converges to $f_0(T_1, T_2, \cdots, T_N)$ if and only if the sequence $\{f_\alpha(T_1, T_2, \cdots, T_N)(x_1, x_2, \cdots, x_N)\}$ converges to $f_0(T_1, T_2, \cdots, T_N)(x_1, x_2, \cdots, x_N)$.

2. Results.

Theorem 1. If a sequence $\{f_n(T_1, T_2, \cdots, T_N)\}$ converges to $f_0(T_1, T_2, \cdots, T_N)$ with respect to the strong operator topology, then spectra $\sigma[f_n(T_1, T_2, \cdots, T_N)]$ converges to $\sigma[f_0(T_1, T_2, \cdots, T_N)]$.

Proof. First of all, we would like to prove that $f_n(\lambda_1, \lambda_2, \cdots, \lambda_N)$ conveges uniformly on $\prod_{k=1}^{N} \sigma(T_k)$. For any element $f_n, f_m \in f \prod_{k=1}^{N} \sigma(T_k)$,

$$f_n(\lambda_1, \lambda_2, \cdots, \lambda_n) - f_m(\lambda_1, \lambda_2, \cdots, \lambda_N) = h_{n,m}(\lambda_1, \lambda_2, \cdots, \lambda_N) \quad (n \neq m)$$

is an element of $\sigma[h_{n,m}(T_1, T_2, \cdots, T_N)]$. For if $h_{n,m}(\lambda_1, \lambda_2, \cdots, \lambda_N) \notin \sigma[h_{n,m}(T_1, T_2, \cdots, T_N)]$, then this contradicts to the fact that $f_n(\lambda_1, \lambda_2, \cdots, \lambda_N) \in \sigma[f_n(T_1, \cdots, T_N)]$. (see [2]). Hence

$$|h_{n,m}(\lambda_1, \lambda_2, \cdots, \lambda_N)| \leq \| h_{n,m}(T_1, T_2, \cdots, T_N) \|.$$

Therefore

$$|f_n(\lambda_1, \lambda_2, \cdots, \lambda_N) - f_m(\lambda_1, \cdots, \lambda_N)| \longrightarrow 0 \text{ as } n, m \longrightarrow \infty,$$

this means that $\{f_n(\lambda_1, \lambda_2, \cdots, \lambda_N)\}$ is a Cauchy sequence in the complex plane, so there is a function $q(\lambda_1, \lambda_2, \cdots, \lambda_N)$ such that

$$f_n(\lambda_1, \lambda_2, \cdots, \lambda_N) \longrightarrow g(\lambda_1, \lambda_2, \cdots, \lambda_N), \text{ as } n \longrightarrow \infty.$$

We have to show

$$g(\lambda_1, \lambda_2, \cdots, \lambda_N) \equiv f_0(\lambda_1, \lambda_2, \cdots, \lambda_N).$$

Let $f_M(\lambda_1, \lambda_2, \cdots, \lambda_N) \in \sigma[f_M(T_1, T_2, \cdots, T_N)]$ for sufficiently large M, so that

$$f_M(T_1, T_2, \cdots, T_N) \in S_{\varepsilon/1}(f_0(T_1, T_2, \cdots, T_N))$$

and $$|f_M(\lambda_1, \lambda_2, \cdots, \lambda_N) - g(\lambda_1, \lambda_2, \cdots, \lambda_N)| < \varepsilon/2.$$

Suppose that

$$g(\lambda_1, \lambda_2, \cdots, \lambda_N) \notin \sigma[f_0(T_1, T_2, \cdots, T_N)].$$

Since
$$\begin{aligned} &\| \{f_n(\lambda_1, \lambda_2, \cdots, \lambda_N)I - f_n(T_1, T_2, \cdots, T_N)\} \\ &- \{g(\lambda_1, \lambda_2, \cdots, \lambda_N)I - f_0(T_1, T_2, \cdots, T_N) \| \\ &\leq |f_n(\lambda_1, \lambda_2, \cdots, \lambda_N)I - g(\lambda_1, \lambda_2, \cdots, \lambda_N)| \\ &\quad + \| f_n(T_1, T_2, \cdots, T_N) - f_o(T_1, T_2, \cdots, T_N) \| < \varepsilon \quad \text{for } n \geq M, \end{aligned}$$

and $[g(\lambda_1, \lambda_2, \cdots, X_N) - f_0(T_1, T_2, \cdots, T_N)]^{-1}$ exists, so does

$$[f_M(\lambda_1, \lambda_2, \cdots \lambda_N) - f_M(T_1, T_2, \cdots, T_N)]^{-1}.$$

But this contradicts the fact that

$$f_M(\gamma_1, \gamma_2, \cdots, \gamma_N) \in \sigma[f_M(T_1, T_2, \cdots, T_N)].$$

Therefore

$$g(\gamma_1, \gamma_2, \cdots, \gamma_N) \notin \sigma[f_0(T_1, T_2, \cdots, T_N)].$$

According to the generalized spectral mapping theorem, any number $\mu \in \sigma[f_0(T_1, T_2, \cdots, T_N)]$ can be represented in the form

$$\mu = f_0(\lambda_1, \lambda_2, \cdots, \lambda_N), \ (\lambda_1, \lambda_2, \cdots, \lambda_N) \in \prod_{k=1}^{N} \sigma(T_k).$$

Thus we can identify $f_0(\lambda_1, \lambda_2, \cdots, \lambda_N)$ and $g(\lambda_1, \lambda_2 \cdots, \lambda_N)$ that is,

$$f_0(\lambda_1, \lambda_2, \cdots, \lambda_N) \equiv g(\lambda_1, \lambda_2, \cdots, \lambda_N).$$

Theorem 2. For an operator $f(T_1, T_2, \cdots, T_N) \in A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X]$ there exists an element $(\zeta_1^{(0)}, \zeta_2^{(0)}, \cdots, \zeta_N^{(0)}) \in \overline{D}(f)$ such that

$$\| f(T_1, T_2, \cdots, T_N) \| = |f(\zeta_1^{(0)}, \zeta_2^{(0)}, \cdots, \zeta_N^{(0)}|$$

Proof.
$$\begin{aligned} \| f(T_1, T_2, \cdots, T_N) \| &= \sup_{\|(X_1, X_2, \cdots, X_N)\| \leq 1} \| f(T_1, T_2, \cdots, T_N)(X_1, X_2, \cdots, X_N) \| \\ &\leq \sup_{\|(X_1, X_2, \cdots, X_N)\| \leq 1,} \ \sup_{(\zeta_1, \zeta_2, \cdots \zeta_N) \in D(f)} |f(\zeta_1, \zeta_2, \cdots, \zeta_N)| \\ &\quad \times \| \frac{1}{2\pi_i} \int_{C_1} R_{\xi_1}(T_1) x_1 d\zeta_1 \| \cdots \| \frac{1}{2\pi_i} \int_{C_N} R_{\xi_N}(T_N) x_N d_{\xi N} \| \end{aligned}$$

$$\leq \sup_{(\zeta_1,\zeta_2,\cdots,\zeta_N)\in D(f)} |f(\zeta_1,\cdots,\zeta_N)| \sup_{1\leq k\leq N} \|x_k\| \cdots \sup_{1\leq k\leq N} \|x_k\|$$

$$\leq \sup_{(\zeta_1,\zeta_2,\cdots,\zeta_N)\in D(f)} |f(\zeta_1,\zeta_2,\cdots,\zeta_N)|$$

since $\frac{1}{2\pi i}\int_C R_\xi(T)d\zeta = I$ and $\sup_{1\leq k\leq N} \|x_k\| \leq 1$.

On the other hand for any $(\lambda_1, \lambda_2, \cdots \lambda_N) \in \prod_{k=1}^{N} \sigma(T_k)$

$$\|f(T_1, T_2, \cdots, T_N)\| \geq |f(\lambda_1, \lambda_2, \cdots, \lambda_N)|$$

i, e $\|f(T_1, T_2, \cdots, T_N)\| \geq \sup_{(\zeta_1,\zeta_2,\cdots,\zeta_N)\in\prod_{k=1}^{N}\sigma(T_k)} |f(\lambda_1, \lambda_2, \cdots, \lambda_N)|$.

Moreover $f(\zeta_1, \zeta_2, \cdots, \zeta_N)$ is holomorphic, so does $|f(\zeta_1, \cdots, \zeta_N)|$. Thus there exists a point $(\zeta_1^{(0)}, \cdots, \zeta_N^{(0)})$ in $\overline{D}(f)$ such that

$$|f(\zeta_1^{(0)}, \zeta_2^{(0)}, \cdots, \zeta_N^{(0)})| = \|f(T_1, T_2, \cdots, T_N)\|$$

since $\prod_{k=1}^{N} \sigma(T_k) \subset \overline{D}(f)$.

Using Theorem 2, I proved the following

Theorem 3. The operator set $A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X]$ forms a Banach algebra.

Proof. Through the simple calculation, we know that the set is a normed linear space. The completeness is proved in the following way:

Let $\{f_n(T_1, T_2, \cdots, T_N)\} \subset A[\prod_{k=1}^{N} \Delta_{\xi k}(T_k), X]$ be a cauchy sequence, that is, for any $\varepsilon > 0$

$$\|f_n(T_1, T_2, \cdots, T_N) - f_m(T_1, T_2, \cdots, T_N)\| < \varepsilon \quad \text{for } n, m \geq M.$$

Thus

$$\sup_{(\lambda_1, \lambda_2, \cdots \lambda_N)\in\prod_{k=1}^{N}\sigma(T_k)} |f_n(\lambda_1, \lambda_2, \cdots, \lambda_N) - f_m(\lambda_1, \lambda_2, \cdots, \lambda_N)| < \varepsilon \quad \text{for } n, m \geq M,$$

hence $|f_n(\lambda_1, \lambda_2, \cdots, \lambda_N) - f_m(\lambda_1, \lambda_2, \cdots \lambda_N)| < \varepsilon$, that is the sequence $\{f_n(\lambda_1, \lambda_2, \cdots, \lambda_N)\}$ is Cauchy in C. Since $\prod_{k=1}^{N} \sigma(T_k)$ is bounded Closed, there exists a holomorphic function $f_0(\lambda_1, \lambda_2, \cdots, \lambda_N)$ such that

$$\sup_{(\lambda_1, \lambda_2, \cdots, \lambda_N)\in\prod_{k=1}^{N}\sigma(T_k)} |f_n(\lambda_1, \lambda_2, \cdots, \lambda_N) - f_0(\lambda_1, \lambda_2, \cdots, \lambda_N)| \leq \varepsilon.$$

Therefore $f_0(\lambda_1, \lambda_2, \cdots, \lambda_N) \in H[\overline{D}]$. Hence, according to the Lemma 2 [1] this corresponds to an operator

$$f_o(T_1, T_2, \cdots, T_N) \in A[\prod_{k=1}^{N} \Delta_t(T_k), X].$$

The other criteria for a Banach algebra are trivial. It remains only **to** show that the inequality

$$\| f(T_1, T_2, \cdots. T_N) \cdot g(T_1, T_2, \cdots, T_N) \| \leq \| f(T_1, T_2, \cdots, T_N) \| \cdot \| g(T_1, T_2, \cdots, T_N)$$

This inequality is however, easily seen from the Theorem 2.

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