On the Structure Space of a Certain Class of Rings

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1. Introduction. Let A be a ring and let $S(A)$ denote the collection of primitive ideals of A. Jacobson [1] defined the closure of a subset T of $S(A)$ as the set of primitive ideals of A containing $D_T = \cap \{P: P \in T\}$. The set $S(A)$ topologized by this closure operator is called the structure space of A. It is well known that if A has an identity element, then $S(A)$ is compact. Now we define $U_a = \{P: P \supsetneqq (a),\ P \in S(A)\}$. Then U_a is an open basis for $S(A)$. A ring A is called a W-ring if every D_{U_a} is modular. W. Lee [2] shows that the structure space of a W-ring is compact if and only if the radical of A is modular. This paper devoted to investigate properties of W-ring and their structure spaces.

2. Basic Properties. We can easily see following Theorems in [1].

Theorem I. The radical of a ring is the intersection of its primtive ideals.

Theorem II. An ideal P of a ring A is primitive if and only if $P = (M: A)$ where M is a modular maximal right ideal.

Theorem III. If B is modular ideal of a ring A and e is a left (right) identity modulo B, then e is a two-sided identity modulo B.

Theorem IV. If B_1 and B_2 are modular ideals of a ring A, then the intersection of B_1 and B_2 is modular.

Theorem V. Let B be an ideal in a ring A and $Q(B)$ the set of primitive ideals of A containing B. Then the correspondence $P \to P/B$ is a homeomorphism of $Q(B)$ onto $S(A/B)$.

Now we define the closure of T, $clT = Q(D_T)$.

3. W-Rings.

Dehinition 1. A ring A is called W-ring if and only if every D_{U_a} is modular.

Lemma 1. Let B be an ideal in a W-ring A. Then A/B is a W-ring.

Proof. Let $Q(B)$ be the set of primitive ideals containing B and Va be the image of $U_a \cap Q(B)$ in $S(A/B)$ by the mapping $P \to P/B$. Then

$$D_{V_a} = \cap \{P/B : P \in U_a \cap Q(B)\} = (\cap \{P : P \in U_a \cap Q(B)\})/B$$
$$\supseteq (\cap \{P : P \in U_a\})/B = D_{U_a}/B.$$

Since D_{U_a} is modular, there is an element e in A as an identity modulo D_{U_a}. Then, for every $\bar{a} \in A/B$, we have $\bar{a} - \bar{e}\bar{a} = (a - ea + B) \in D_{U_a}/B \subseteq D_{V_a}$ where $\bar{a} = a + B, \bar{e} = e + B$ and $a, e \in A$. Similarly $\bar{a} - \bar{a}\bar{e} \in D_{V_a}$. This implies that $e + B$ is an identity modulo D_{V_a}, and thus D_{V_a} is modular in A/B. Therefore A/B is a W-ring.

Theorem 1. Let R be the radical of A. Then A is a W-ring if and only if A/R is a W-ring.

Proof. Since R is an ideal in a W-ring A, A/R is a W-ring by Lemma 1. Now suppose A/R is a W-ring. Since any primitive ideal in A contains the radical R of A, S(A) is homeomorphic to $S(A/R)$ by Theorem V. Let Va be the set of P/R where $P \in U_a$. Then $D_{V_a} = \cap \{P/R : P \in Ua\} = (\cap \{P : P \in U_a\})/R = D_{U_a}/R$. Since D_{V_a} is modular, there is an identity $\bar{e}$ in A/R modulo D_{V_a} such that $\bar{a} - \bar{e}\bar{a}$ and $\bar{a} - \bar{a}\bar{e}$ belong to D_{U_a}/R for all $\bar{a}$ in A/R, where $\bar{a} = a + R$, $\bar{e} = e + R$ and a, e in A. Thus $a - ea$ and $a - ae$ belong to D_{U_a} for every a in A. Therefore D_{U_a} is modular in A and this proves that A is a W-ring.

4. The Structure Space of a W-Ring.

Lemma 2. If B is a modular ideal of a ring A, then $S(A/B)$ is compact.

Proof. Since B is modular, there is an element e in A such that $a - ae$ and $a - ea$ belong to B for all a in A. It follows that A/B is a ring with an identity $e + B$ and thus $S(A/B)$ is compact.

Corllary. If P is any primitive ideal in a W-ring, then $S(A/P)$ is compact.

Theorem 2. If A is a W-ring, then $S(A)$ is locally compact. [2]

PROOF. Let P be a point of $S(A)$ and let U_a be the open neighourhood of P. Since $clU_a = Q(D_{U_a})$ and D_{U_a} is modular. $S(A/D_{U_a})$ is compact by **Lemma 2.** Threrfore its homeomorphic image clU_a is compact and thus $S(A)$ is locally compact.

Theorem 3. Let A be a W-ring and F any closed subset of $S(A)$. Then F is compact if and only if D_F is modular.

Proof. Since F is closed set. $F = clF = Q(D_F)$. Hence assume that D_F is modular. Then F is compact by lemma 2. Now suppose that F is compact. Since $\{U_a\}_a \in_A$ is an open cover of F, there is a finite subcover $\{U_{a_n}\}$ $(n = 1, 2, \cdots, m)$. Then $D \cup {}^m{}_{n=1} U_{a_n} \subseteq D_F$ since $\cup {}^m{}_{n=1} U_{a_n} \supseteq F$. On the other hand, $D \cup {}^m{}_{n=1} U_{a_n} = D_{U_{a1}} \cap D_{U_{a2}} \cap \cdots\cdots \cap D_{U_{am}}$. Therefore $D \cup {}^m{}_{n=1} U_{a_n}$ is modular by theorem IV and thus D_F is modular.

References

1. N. Jacobson, *Structure of Rings*, Vol. 37, Amer. Math. Soc. Providenc, 1956
2. W. Lee, *On the compactness of structure space of a ring*, Proc. Amer. Math. Soc. (to appear)

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