A Note on the Convex Set

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The existence of solutions of many optimization problems has a close connection with the closure of some convex sets in an n-dimensional Euclidean space E^n. Suppose that X is a bounded, non-empty convex set in E^n. we define $H(p)$ to be the supporting hyperplane of X which is normal to any nonzero vector p in E^n and put $G(p) = H(p) \cap X$.

It is the purpose of this note to derive a necessary and sufficient condition for set X to be closed without the convexity conditions of the sets $G(p)$. We define the distance between two sets $G(p)$ and $G(q)$ as follows:

$$d(G(p),\ G(q)) = \inf\ \{d(x, y)\,|\,x \in G(p), y \in G(q)\}.$$

Theorem. A necessary and sufficient condition for the bounded, non-empty convex set X to be closed is that

(1) for every nonzero vector p the set $G(p)$ is closed and non-empty;

(2) if $p_1, p_2, \cdots$ is a sequence of nonzero vectors converging to some nonzero vector p then

$$\lim_{i\to\infty} d(G(p_i),\ G(p)) = 0.$$

Proof. The necessary condition: The proof of (1) is obvious. The proof of (2) is as follows: Since the set $G(p_i)$ and $G(p)$ are closed and bounded, there exists a_i which is a point in $G(p_i)$ nearest to the set $G(p)$ for every $i = 1, 2, \cdots$ Let $d_i = d(a_i, G(p))$. Assuming that there is a $\rho > 0$ and a subsequence $k_1, k_2, \cdots$ of $1, 2, \cdots$ such that $d_{k_i} \geqq \rho$ for all $i = 1, 2, \cdots$, it has a subsequence $a_{n_1}, a_{n_2} \cdots$ converging to some point $a \bar{\in} G(p)$, because the sequence of points $a_{k_1}, a_{k_2} \cdots$ is bounded. For every $x \in X$ we have the scalar product of two vectors a and p

$$a \cdot p = \lim_{i\to\infty}(a_{n_i} \cdot p_{n_i}) \geqq \lim_{i\to\infty}(x \cdot p_{n_i}) = x \cdot p$$

which implies that $a \in H(p)$ and contradicts the assumption $a \bar{\in} G(p)$. Thus we have $\underset{i\to\infty}{\text{him}}$ $(G(p_i), G(p))=0$.

The sufficient condition: Let $a \in X^d$. We know $a \in H(p)$ for some non-zero vector p. If $a \in G(p)$ let b be the point oi $G(p)$ nearest to a. Since the set $G(p)$ is closed and $a \neq b$, there exists an $\varepsilon > 0$ such that $d(G(p), G(p_\varepsilon)) \leqq |a-b|/2$ where $p_\varepsilon = p + \varepsilon(a-b)$. Let $x \in G(p_\varepsilon)$ and $y \in G(p)$ such that $|x-y| \leqq |a-b|/2$. We have $p_\varepsilon \cdot a \leqq p_\varepsilon \cdot x$ and $p \cdot x \leqq p \cdot a$; hence$(a-b)\cdot(x-a) \geqq 0$. Since the vector $a-b$ is an outward normal to a supporting hyperplane of the convex set $G(p)$ at the point b and since $y \in G(p)$, we have$(a-b)\cdot(b-y) \geqq 0$. From the last two inequalities we obtain $|x-y| \geqq |a-b|$ which contradicts the relation $|x-y| \leqq |a-b|/2$. Consequently if $a \in X^d$, then $a \in X$.

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