On the Studies of the Total Absolute Curvature of Immersed Manifolds

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This is an abstract of a talk about the studies of the total absolute curvature of immersed manifolds including some of recent problems unsolved introduced by the author at the meeting of the Korean Mathematical Society held at Pusan University, Pusan, Octover 25-26, 1968. The author expresses his thanks to Prof. T. J. Willmore, The University of Durham, England, to whom he has owed in his study of this direction.

Let f: $M^n \to E^m$ be a smooth mapping of a smooth n-manifold M^n into an Euclidean space of dimension m. We call the pair (M^n, f), or simply f, an immersed manifold of M^n if the differential map $df_p: T_p(M^n) \to E^m$ on the tangent space to M^n at p for every point p in M^n has the maximal rank and the pair (M^n, f), or simply f, an embedded manifold of M^n if in addition f: $M^n \to E^m$ is homeomorphic (into). In particular if M^n is compact, an embedded manifold requires the additional condition of f: $M^n \to E^m$ to be injective only.

By S. S. Chern and R. K. Lashof [1], the total absolute curvature $\tau(M^n, f, E^{n+N})$ of an immersed manifold (M^n, f) is defined by

(1) $$\tau(M^n, f, E^{n+N}) = \int_{p \in M^n} K^*(p)\, dv,$$ if it exists, where

(2) $$K^*(p) = \int_{F(p)} |G(p, v)|\, d\delta_{N-1},$$

where $F(p)$ is the fiber at p in M^n of unit normal bundle B_ν over M^n, and dv and $d\delta_{N-1}$ are the volume elements of M^n and $(N-1)$-sphere for each fiber respectively, and $G(p, v) = G(p, v(p))$ is the Lipschitz-Killing curvature at $(p, v(p))$ in B_ν, which is a generalization of the Gaussian curvature on a surface or Gauss-Kroneker curvature of n-dim. Riemannian surface, defined by

(3) $\nu^* d\Sigma = G(p, v)\, d\delta_{N-1}\, dv$, where ν^* is the dual mapping on the differ-

ential forms induced by the smooth mapping $\nu: B \to {}_{\nu}S_0^{n+N-1}$ of B_2 into the $(n+N-1)$ sphere in E^{n+N} which assigns to each point $(p, v(p))$ of B a point z of S_0^{n+N-1} paralell as a vector to the unit normal vector $v(p)$ at $f(p)$ in $f(M^n)$, and $d\Sigma$ is the volume element of S_0^{n+N-1}.

The results of Chern and Lashof in [1] are with the assumption that M^n is a compact oriented smooth manifold immersed in E^{n+N} as follows:

Theorem 1. $\tau(M_n, f, E^{n+N}) \geq 2C_{n+N-1}$ for any immersion $f: M^n \to E^{n+N}$.

Theorem 2. $\tau(M^n, f, E^{n+N}) < 3C_{n+N-1}$ implies that M^n is homeomorphic to a sphere of n-dimension.

Theorem 3. $\tau(M^n, f, E^{n+N}) = 2C_{n+N}$ implies that M^n belongs to a linear subvariety E^{n+1} of dimension $n+1$, and is embedded as a convex hypersurface in E^{n+1}. The converse is also true, where $C_{n+N-1} = \int_{S_0^{n+N-1}} d\Sigma$.

By N. H. Kuiper later on, the total absolute curvature in a normalized form,

$$(4) \qquad \tau(f) = \tau(M^n, f, E^{n+N}) = \int_{B_\nu} \frac{|\nu^*(d\Sigma)|}{C_{n+N\ 1}},$$

is investigated through the application of M. Morse theory, after constructing smooth functions on M^n[3]. Thus the studies of M. Morse theory will be very important in this direction. M. Morse theory appeared in [10] and exposed systematically by J. Milnor [11].

Let $\beta_k(M^n)$ be the maximal rank of the k-th homology group of M^n, for all coefficient rings and the class Φ of those real differentiable functions $\varphi: M^n \to R$, which have no other than nonsingular, hence isolated, critical points. Let $\mu_k(M^n, \varphi)$ be the number of critical points of index k of φ. A critical point is of index k if in some neighborhood of this point

$$(\varphi_1, \varphi_2, \cdots, \varphi_n) = (0, 0 \cdots, 0),$$

$$(5) \qquad \varphi = C - \varphi^2_1 - \cdots - \varphi^2_k + \varphi^2_{k+1} + \cdots + \varphi^2_n$$

in local coordinates $\varphi_1, \cdots, \varphi_n$, where C is some constant.

Moreover we define

$$(6) \qquad \mu_k(M^n) = \min_{\varphi \in \Phi} \mu_k(M^n, \varphi)$$

(7) $$\mu(M^n) = \min_{\varphi\in\Phi} \sum_{k=0}^{n} \mu_k(M^n, \varphi),$$

and also

$\mu(M^n, \varphi) =$ the number of isolated critical points of φ in case $\varphi\in\Phi$.

Then, the Morse inequalities [11] are

(8) $$\mu_k(M^n) \geq \beta_k(M^n)$$

(9) $$\mu(M^n) \geq \sum_{k=0}^{n} \mu_k(M^n) \geq \sum_{k=0}^{n} \beta_k(M^n) = \beta(M^n).$$

In particular for the real projective space P^n,

$$\mu_k(P^n) = \beta_k(P^n) = 1$$

and

$$\mu(P^n) = \beta(P^n) = n + 1.$$

Some important functions on an immersed manifold as members of the family Φ are the following:

Let $f, M^n, E^{n+N}, B_\nu, S_0^{n+N-1}$ and $\tilde{\nu}$ be as before. A point $z\in S_0^{n+N-1}$ is defined as a critical value of f if there exists a point $b\in B_\nu$ with $\tilde{\nu}(b) = z$ and $\tilde{\nu}$ has rank $< n + N - 1$ at b. M^n being compact, the set of critical values of f is a closed set in S_0^{n+N-1}. According to the theorem of Sard [12] the set of critical values of f in S_0^{n+N-1} has measure zero. Let G be the open set of non-critical values of f in S_0^{n+N-1}, and consider a non-critical value $z\in G\subset S_0^{n+N-1}\subset E^{n+N}$. Now we consider E^{n+N} again as an Euclidean vector space with variable point y.

The inner product zy is a linear function in y, i.e.

$$\phi\colon E^{n+N} \longrightarrow R \quad \phi y \quad \phi(y) = zy$$

and the restriction of ϕ to $f(M^n)$, denoted again by ϕ, is $\phi(f(m)) = zf(m)$. Thus we have a function $\varphi\colon M^n \to R$ defined by $\varphi(m) = \phi(f(m)) = zf(m)$ for each $z\in G\subset S_0^{n+N-1}$. This function $\varphi = zf\colon M^n \to R$ has only non-degenerate isolated critical points in M^n and hence $\varphi = zf\in\Phi$. $\mu(M^n, zf)$ is a function of $z\in S_0^{n+N-1}$.

With the above preparation, Kuiper [3] expressed $\tau(M^n, f, E^{n+N})$ by

(10) $$\tau(M^n, f, E^{n+N}) = \int_{B_\nu} \frac{|\nu^* d\Sigma|}{C_{n+N-1}} = \int_{\nu^{-1}(G)} \frac{|\nu^* d\Sigma|}{C_{n+N-1}}$$

$$=\int_{z\in G}\frac{\mu(M^n, zf)\,|d\Sigma|}{C_{n+N-1}}$$

$$=\int_{z\in S_0{}^{n+N-1}}\frac{\mu(M^n, zf)\,|d\Sigma|}{C_{n+N-1}}.$$

Since $\mu(M^n, zf)\geq\mu(M^n)$, we have

(11) $$\tau(M^n, f, E^{n+N})\geq\int_{z\in S_0{}^{n+N-1}}\frac{\mu(M^n)\,|d\Sigma|}{C_{n+N-1}}=\mu(M^n).$$

Some important results obtained by Kuiper [3] are as follows:

Theorem 4. If the compact n-manifold M^n is fixed, then for variable immersion $f: M^n\to E^{n+N}$ and variable N:

(12) $$inf.\ \tau(M^n, f, E^{n+N})=\mu(M^n).$$

Theorem 5. For any immersion of any compact manifold M^n.

(13) $$\mu(M^n, f, E^{n+N})\geq\mu(M^n)\geq\beta(M^n)\geq 2.$$

We can easily notice That the theorem 5 is sharper than the Theorem 1.

Some of significant terminologies in these studies are as follows:

Definition. An immersion f: $M^n\to E^{n+N}$ is called proper into E^{n+N} if $f(M^n)$ is not contained in any hyperplane E^{n+N-1} in E^{n+N}

Definition. An immersion $f: M^n\to E^{n+N}$ is called tight if the *inf* of $\tau(f)$ is attained.

With a property such that for any real function φ with isolated singularities on a compact connected manifold M^n for which $\mu(M^n,\varphi)=\mu(M^n)$, it is true that $\mu_0(M^n,\varphi)=\mu_n(M^n,\varphi)=1$. Kuiper [3] has proved the following:

Theorem 6. For the positive integer $N\leq\frac{1}{2}n(n+1)$, there exists a manifold M^n and a tight immersion $f: M^n\to E^{n+N}$ of M^n proper into E^{n+N}.

As a specific example of tight and proper immersions Kuiper proved in [3]:

Theorem 7. There exists a tight imbedding $f: P^2\to E^4$ of real projective plane P^2 proper into E^4.

In his paper [5] appeared at the same time as [2], Kuiper proved:

Theorem 9. If f is a tight immersion of torus T in E^3 then $f(T)$ has two particular tangent planes which meet $f(T)$ in two pointsets that divide the complement in a first part, all point of which have non-negative total curvature and which are containd in the boundary of the convex hull of $f(T)$ and a second part, all points of which have non-positive total curvature.

In the proofs of the above theorem, he made use of some results from the analysis of top sets, which seems to play important rôle in the studies of this area.

If V is a set in an Euclidean (vector) space E^n, then the smallest closed convex body which contains V is called the convex hull of V denote by $h(V)$. The intersection of $h(V)$ and the closure of its complent in E^n is the boundary $\partial h(V)$, which is called convex envelope of V in E^n.

If η is any linear function on E^n, for instance, the inner product with a fixed unit vector $z \in E^n$, which assigns to the vector $y \in E^n$ the value zy, then the hyperplane consisting of all points in E^n for which η assumes a value equal to its maximum on $h(V)$, is called the top hyperplane of the function.

The intersection of a top hyperplane of the function η and V is called a top set of V in the direction z, denoted by $\Delta(z, V)$, when z is chosen to define the function $\eta = yz$.

For the mapping $f: M \to E$ the word top set is also used for the corresponding set in M, consisting of all $m \in M$ with $zf(m) = \max_{x \in M}(zf(x))$, and this top set is denoted by $\Delta(z, f, m)$.

Theorem 10. If $f: M \to E^3$ is tight, with M compact, then convex envelope $\partial h(f(M))$ contains all points of $f(M)$ with $K > 0$ and no point with $K < 0$. If curvature K at $m \in M$ is postive then $f(m) \in f(M) \cap \partial h(f(M)) \subset \partial h(f(M))$.

Examples of tight surfaces in E^3:

(a) $\chi = 2$: the surface of any convex body in E^3.

(b) $\chi = 0$: the torus obtained by rotating circle about a non-intersecting line in the same plane.

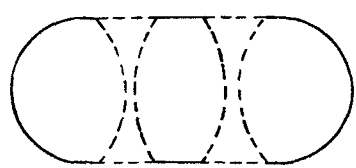

Fig. of $ex(c)$ for $g=2$

(c) $\chi = 2\text{-}2g$: the sphere with $g(\geq 0)$ handles which give no contribution to the parts of $K>0$.

We will consider a reduction of the total absolute curvature of immersed manifolds to the case of $n=2$ and $N=1$.

Let M be a compact two-dimensional surface. The Euler-Poincare characteristic $\chi(M)$ of M can be defined thnough a triangulation of M. If e_0, e_1, and e_2 are the numbers of vertices, edges and triangles of the triangulation respectively, then

(14) $$\chi(M) = \sum_k (-1)^k e_k.$$

If $H_k(M, F)$ is the k-dimensional homology groups of M with respect to any field F, then

(15) $$\chi(M) = \sum_k (-1)^k \dim \mathrm{H}_k(M, F).$$

Let φ be a differentiable function on M, then

(16) $$\chi(M) = \sum_k (-)^k \mu_k(M, \varphi),$$

where $\mu_k(M, \varphi)$ is the number of critical points of index k. In (14), (15) and (16), the expressions are independent of the triangulation, the coefficient field, and the differntiable function, respectively.

A complete classification of compact surfaces which applies to topological as well as to differentiable structures, is obtained by the orientation and the characteristic χ together. An orientable surface is a sphere with $g(\geq 0)$ handles and has $\chi = 2\text{-}2g$. A non-orientable surface is a projective plane or a Klein-bottle with $g(\geq 0)$ handles and has $\chi = 1-2g$ or $\chi = -2g$ respectively.

Suppose the compact surface M has a Riemannian metric. Let K be the Gaussian curvature. Then the well known Gauss-Bonnet theorem says:

(17) $$\chi(M) = \int_M \frac{K}{2\pi}\, d\mu,$$

where $d\mu$ is an exterior 2-form as the volume element of M. This can be proved easily making use of (14).

In his paper [5], Kuiper proved the Gauss-Bonnet theorem for more general case of an abstract Riemannian surface M given as an immersion $f: M \to E^3$, by applying (16) which comes from the theory of M. Morse.

Now let consider the total absolute curvature $\tau(M) = \tau(M, f, E^3)$ of immersed manifold (M, f) in E^3. By its definition,

$$\tau(M) = \tau(M, f, E) = \int_M \frac{|K|}{2\pi} d\mu = \int_{K>0} \frac{Kd\mu}{2\pi} - \int_{K<0} \frac{Kd\mu}{2\pi}, \tag{18}$$

where by $k > 0$, we mean the open set of all $m \in M$ at which $k > 0$. By the generalized Gauss-Bonnet theorem,

$$\chi(M) = \int_M \frac{Kd\mu}{2\pi} = \int_{K>0} \frac{Kd\mu}{2\pi} + \int_{K<0} \frac{Kd\mu}{2\pi}.$$

Hence

$$\begin{aligned} |\chi(M)| &= \left| \int_{K>0} \frac{Kd\mu}{2\pi} + \int_{K<0} \frac{Kd\mu}{2\pi} \right| \\ &= \left| \int_{K>0} \frac{Kd\mu}{2\pi} \right| + \left| \int_{K<0} \frac{Kd\mu}{2\pi} \right| \\ &= \int_{K>0} \frac{Kd\mu}{2\pi} - \int_{K<0} \frac{Kd\mu}{2\pi} = \tau(M). \end{aligned}$$

Thus we have:

Theorem 11. The total absolute Gauss-curvature of an abstract compact surface M with Riemannian metric and not neccesarily orientable obeys:

$$\tau(M) = \int_M \frac{|K| d\mu}{2} \geq |\chi(M)|. \tag{19}$$

On the other hand we have due to kuiper [5] the following:

Theorem 12. The total absolute Gauss-curvature of a compact smooth surface M in Euclidean 3-space E^3 (not neccesarily orientable) obeys:

$$\tau(M) = \int_M \frac{|K| d\mu}{2\pi} \geq 4 - \chi(M). \tag{20}$$

Only for the case $\chi(M) = 2$ this lower bound is not higher than the lower bound given in (19) since $\chi(M)$ is integer and ≥ 2. It follws that every topological compact surface M, for which $\chi(M) \neq 2$, admits Riemannian metric which can not be realized by a smooth immersion in E^3. In particular the elliptic plane (projective plane with $K = 1$), the locally Euclidian surface ($K = 0$) and the compact surfaces of constant negative curvature

$(K=-1)$ are surfaces with Riemannian metric that cannot be obtained as surfaces in E^3.

By definition, if an immersion $f: M \to E^3$ of a compact surface M is convex, we have

$$(21) \qquad \tau(M) = \int_M \frac{|K|\,d\mu}{2} = 4 - \chi(M).$$

In the above example (c) of the tight immersion E^3, it follows that

$$(22) \qquad \tau(M) = \tau(M, f, E^3) = 4 - \chi(M) = 4-(2-2g) = 2+2g.$$

This says for any given positive even integer we can find a immersion in E^3 of which total absolute curvature is the integer.

A new direction of the studies of this area has been studied by T. J. Willmore [7], which is to replace the Euclidean space E by an Riemannian manifold M' in the immersion $f: M^n \to E^{n+N}$ and investigate the corresponding theorems. Another new direction seems to generalize the definition of $\tau(M, f, E)$ to the case where M is an infinite dimensional smooth manfiold and E is a Banach space. The author has felt such a motivation in the work of J. Eells Jr. and J. H. Sampson [8].

A lot of unsolved interesting problems in this area have been given by those working on it, for instance, S. S. Chern, N. H. Kuiper, T. J. Willmore, T. F. Banchoff and etc. More problems has been appeared in N. H. Kuiper's recent paper. [6].

Among others, some problems are as follows:

1) Does there exists a tight immersion of the real projective plane with one handle in E^3, i. e, immersion such that $\tau(M, f, E^3) = \frac{1}{2\pi}\int |K|\,d\mu = 5$?

2) A necessary condition for M^2 to be a tight immersion into E^3 is

$$\tau(E^2, f, E^3) = \frac{1}{2\pi}\int |K|\,d\mu = 4 - \chi(M^2).$$

Are there any further necessary conditions?

3) Rigidity problem: According to Hadamard theorem two isometrice convex closed surfacse are congruent. Suppose $f, g: M^2 \to E^3$ are tight, does there exists any isometry between $f(M^2)$ and $g(M^2)$ for general surfaces?

4) By Banchoff's theorem [9], for an $f: M^n \to E^{n+N}$, a tight and proper immersion of C^0-manifold M^n, we have

$$N \leq \frac{1}{2}(7 + \sqrt{49 - 24\chi})$$

What can be said about C^1-manifolds?

5) Theorem 5 says there exists a tight and proper immersion $f: P^2 \to E^4$ of real projective plane P^2. What can be said about M^2 for which (M^2, f) are tight and proper immersed manifolds in E^4?

6) When a E^{n+N} is replaced by a Riemannian manifold M' in an immersion $f: M^n \to E^{n+N}$, how further [7] corresponding theorems shall hold?

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