On Semi-simplicity and Weak Semi-simplicity

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It is well known that the radical $J(A)$ of a ring A is the intersection of the modular maximal right ideals of A and a ring A is semi-simple if and only if $J(A) = (0)$. The purpose of this note is to prove that a semi-simple ring is weakly semi-simple and that if A is a commutative ring, then (0) is almost maximal if and only if A is an integral domain. The notation and terminology of this note are based on [1].

DEFINITION 1. If I is a proper right ideal of a ring A then I is *almost maximal* provided that

(1) if J_1 and J_2 are right ideals of A and $J_1 \cap J_2 = I$, then $J_1 = I$ or $J_2 = I$, i.e., I is meet-irreducible,

(2) if $a \in A$ and $[I:a] \supset I$, then $a \in I$,

(3) if J is a right ideal of A, $J \supset I$, then $N(I) \cap J \supset I$, where $N(I) = \{a \in A: aI \subseteq I\}$, and if $a \in A$ such that $[J:a] \supseteq I$ then $[J:a] \supset I$.

THEOREM 1. *If A is a commutative ring, then* (0) *is an almost maximal ideal if and only if A is an integral domain.*

Proof. Let b be a nonzero element of A and $ab = 0$ for some $a \in A$. Then $[(0):a] \ni b$, i.e., $[(0):a] \supset (0)$. Since (0) is almost maximal, $a \in (0)$, i.e., $a = 0$. Hence A is an integral domain. Conversely, if J_1 and J_2 are right ideals of A, $J_1 \supset (0)$ and $J_2 \supset (0)$, then there exist nonzero elements a in J_1 and b in J_2 such that $ab \neq 0$, since A is an integral domain, and $ab \in J_1 \cap J_2$, i.e., $J_1 \cap J_2 \supset (0)$. Now suppose $a \in A$ and $[(0):a] \supset (0)$, then there exists a nonzero element b in A such that $ab = 0$. Since A is an integral domain, $a = 0$, i.e., $a \in (0)$. Again assume $J \supset (0)$. Since $N((0)) = A$, $J \cap N((0)) = J \supset (0)$. And if $a \in A$, then $[J:a] \supseteq J \supset (0)$. Therefore (0) is almost maximal.

The proofs of Theorem 2 and Corollary depend upon the following lemmas proved in [1].

LEMMA 1. *Let $W(A)$ be the weak radical of a ring A. If $W(A) \neq A$, then $W(A)$ is the intersection of almost maximal right ideals of A.*

LEMMA 2. *If A is a ring and B is a two-sided ideal of A, then $W(B) = W(A) \cap B$.*

DEFINITION 2. A ring is called *weakly semi-simple* if and only if its weak radical is zero ideal.

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THEOREM 2. *If a ring A is semi-simple, then A is weakly semi-simple.*

Proof. We show that a modular maximal right ideal I of A is almost maxial, from which it follows that $W(A)\subseteq J(A)$ by Lemma 1, and the theorem will be proved. Assume J is a right ideal of A and $J\supset I$. Then $J=A$ since I is maximal. Therefore I is meet-irreducible in A. Now suppose $a\in A$ and $[I:a]\supset I$. Then $[I:a]=A$, i.e., $aA\subseteq I$. If $a\notin I$, then $a+I$ is a generator of a strictly cyclic A-module $A-I$. For any b in A, there exists an element c in A such that $(a+I)c=b+I$. Then $b\in I$, since $ac-b\in I$ and $ac\in I$, which is impossible. Hence $a\in I$. Let e be a left identity modulo I. Then $I\subseteq N(I)$, $e\in N(I)$ and $e\notin I$, i.e., $N(I)\supset I$. If $J\supset I$, then $J\cap N(I)=N(I)\supset I$. Assume $J\supset I$ and $a\in A$ such that $[J:a]\supseteq I$. Since $[J:a]=[A:a]$, $[J:a]\ni e$ and thus $[J:a]\supset I$. Thus the theorem is proved.

COROLLARY. *Let A be a ring and B be an ideal of A such that A and B have the same radical. Then A and B have the same weak radical.*

Proof. Since $J(A)=J(B)=J(A)\cap B$, $J(A)\subseteq B$ and thus $W(A)\subseteq B$. Therefore, by Lemma 2, $W(B)=W(A)\cap B=W(A)$.

Reference

1. K. Koh and A. C. Newborn, *The weak radical of a ring*, Proc. Amer. Math. Soc. **18** (1967), 554–559.

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